

On the complexity of modal axiomatisations over many-dimensional structures

Agi Kurucz

*Department of Computer Science
King's College London, U.K.
agi.kurucz@kcl.ac.uk*

Abstract

We show that all the complexities of a possible axiomatisation of $\mathbf{S5}^n$, the n -modal logic of products of n equivalence frames, are already present in any axiomatisation of \mathbf{K}^n . Then we show that if $3 \leq n < \omega$ then, for any set L of n -modal formulas between \mathbf{K}^n and $\mathbf{S5}^n$, the class of all frames for L is not closed under ultraproducts and is therefore not elementary. So any modal axiomatisation for a Kripke complete logic in the interval between \mathbf{K}^n and $\mathbf{S5}^n$ must contain modal formulas with no first-order correspondents. The proof is based on a construction of Hirsch and Hodkinson [15] showing that the class of strongly representable n -dimensional cylindric algebra atom structures is not closed under ultraproducts. We show that this construction can be carried through in a diagonal-free setting.

Keywords: many-dimensional modal logic, products of Kripke frames, ultraproducts

1 Introduction

As usual in any area of logic, when one considers the “logic” or “theory” of a class \mathcal{C} of structures (the “intended models”), then there are always “non-intended”, “non-standard” models of this “logic”. These non-standard structures are often hard to describe. In this paper we discuss this problem in the setting of *n-modal logics*: propositional multi-modal logics having finitely many unary modal operators $\diamond_0, \dots, \diamond_{n-1}$ (and their duals $\square_0, \dots, \square_{n-1}$), where n is a non-zero natural number. Formulas of this language, using propositional variables from some fixed countably infinite set, are called *n-modal formulas*. Frames for n -modal logics — *n-frames* — are structures of the form $\mathfrak{F} = (W, T_i)_{i < n}$ where W is a non-empty set and each T_i is a binary relation on W , for $i < n$. *Validity* of a set Σ of n -modal formulas in an n -frame \mathfrak{F} (in symbols: $\mathfrak{F} \models \Sigma$) is defined as usual. If $\mathfrak{F} \models \Sigma$ then we also say that \mathfrak{F} is a *frame for* Σ . Given a class \mathcal{C} of n -frames, we denote by $\text{Log}(\mathcal{C})$ the set of all n -modal formulas that are valid in every n -frame in \mathcal{C} .

Our “intended” structures are the following special n -frames. Given 1-frames $\mathfrak{F}_i = (W_i, R_i)$, $i < n$, their *product* is the n -frame

$$\mathfrak{F}_0 \times \dots \times \mathfrak{F}_{n-1} = (W_0 \times \dots \times W_{n-1}, \bar{R}_i)_{i < n},$$

where $W_0 \times \cdots \times W_{n-1}$ is the Cartesian product of the W_i and for all $\mathbf{u}, \mathbf{v} \in W_0 \times \cdots \times W_{n-1}$ and $i < n$,

$$\mathbf{u}\bar{R}_i\mathbf{v} \quad \text{iff} \quad u_i R_i v_i \text{ and } u_j = v_j \text{ for } j \neq i, j < n.$$

Such n -frames we call *n -dimensional product frames*. They have been introduced in [9,24] and have been extensively studied both in pure modal logic and in applications, see [8,21] and the references therein.

Two examples of classes of n -dimensional product frames are:

$$\begin{aligned} \mathcal{C}_{all}^n &= \text{the class of all } n\text{-dimensional product frames,} \\ \mathcal{C}_{equiv}^n &= \text{the class of all } n\text{-dimensional products of equivalence frames.} \end{aligned}$$

Let us also introduce notations for the n -modal logics they determine:

$$\begin{aligned} \mathbf{K}^n &= \text{Log}(\mathcal{C}_{all}^n), \\ \mathbf{S5}^n &= \text{Log}(\mathcal{C}_{equiv}^n). \end{aligned}$$

It can be hard to describe an arbitrary n -frame for \mathbf{K}^n or $\mathbf{S5}^n$. As is shown in [16], if $n \geq 3$ and L is any set of n -modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$, then it is undecidable whether a finite n -frame is a frame for L or not. (So no such logic L can be finitely axiomatisable.) Here we show that these non-standard n -frames are hard to “catch” in an other sense: They cannot be described in the first-order “frame language”, that is, in the language having n binary predicate symbols and equality.

Theorem 1.1 *Let $3 \leq n < \omega$ and let L be any set of n -modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then the class of all frames for L is not closed under ultraproducts, and so is not elementary.*

Note that both \mathbf{K}^2 and $\mathbf{S5}^2$ are (finitely) axiomatisable by Sahlqvist-formulas (see [9,14]), so the respective classes of all their frames are elementary. Also note that Theorem 1.1 only says that the class of all frames for certain modal logics is not closed under ultraproducts. Such a logic can still be determined by some smaller, ultraproduct-closed class of n -frames. This is indeed the case for many, see Prop. 2.9 below. As is shown in [20], \mathbf{K}^n is even determined by a class of n -frames that can be finitely axiomatised in the first-order frame language.

However, as a consequence of Theorem 1.1 we obtain the following quite discouraging result, as far as finding an explicit axiomatisation for the logics in question is concerned:

Corollary 1.2 *Let $3 \leq n < \omega$ and let L be any Kripke-complete n -modal logic such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then any axiomatisation for L must contain n -modal formulas with no first-order correspondents.*

We conjecture that, for canonical logics L in the interval between \mathbf{K}^n and $\mathbf{S5}^n$, a combination of the techniques of the present paper with those of Hodkinson and Venema [17] might result in an even stronger statement: Any axiomatisation for such an L must contain infinitely many non-canonical n -modal formulas.

The structure of the paper is as follows. In Section 2 we give a general characterisation of arbitrary frames of multi-modal logics determined by frame-classes satisfying some closure conditions. Using this we show that if we could “deal” with non-standard n -frames for \mathbf{K}^n , then we could do that with arbitrary n -frames for $\mathbf{S5}^n$ as well. In particular, we show that $\mathbf{S5}^n$ is finitely axiomatisable over \mathbf{K}^n . Then in Sections 3 and 4 we prove Theorem 1.1. The proof is based on a construction of Hirsch and Hodkinson [15] showing that the class of strongly representable n -dimensional cylindric algebra atom structures is not closed under ultraproducts. We show that this construction can be carried through in a diagonal-free setting, and then apply the results of Section 2.

2 Non-standard frames for logics determined by classes of n -dimensional product frames

We begin with proving some general results on modal logics determined by special classes of relational structures of *any* signature. In what follows we use the words *frame* and *relational structure* as synonyms. (So the n -frames introduced in Section 1 are special frames.) We use without explicit reference standard notions and results from basic modal logic and universal algebra; such as *p-morphisms*, *generated subframes*, *Sahlqvist formulas* and *canonicity*, duality between relational structures and *Boolean algebras with operators (BAOs)*, *homomorphisms*, *subalgebras*, *direct products*, *ultraproducts*, *varieties*, *subdirect embeddings* and *subdirectly irreducible algebras*. For notions and statements not defined or proved here, see [3,4,10,13].

If x is a point in a relational structure \mathfrak{F} then we denote by \mathfrak{F}^x the smallest generated subframe of \mathfrak{F} containing x . We call \mathfrak{F}^x a *point-generated subframe* of \mathfrak{F} . If $\mathfrak{F} = \mathfrak{F}^x$ for some x , then \mathfrak{F} is called *rooted*. Apart from the usual operators \mathbf{H} , \mathbf{S} and \mathbf{P} on classes of algebras (denoting homomorphic images, subalgebras, and isomorphic copies of direct products, respectively), we use the following operators on classes of frames of the same signature:

$\mathbb{G}\text{sf } \mathcal{C}$ = isomorphic copies of generated subframes of frames in \mathcal{C} ,

$\mathbb{G}\text{sf}_p \mathcal{C}$ = isomorphic copies of point-generated subframes of frames in \mathcal{C} .

The *(full) complex algebra* of a frame $\mathfrak{F} = (W, R_i)_{i \in I}$ is denoted by $\mathbf{Cm } \mathfrak{F}$. That is, $\mathbf{Cm } \mathfrak{F} = (\mathcal{P}(W), \cap, -^W, f_i)_{i \in I}$, where $(\mathcal{P}(W), \cap, -^W)$ is the Boolean algebra of all subsets of W , and for each $k + 1$ -ary relation R_i , f_i is a k -ary function defined by taking, for every $X_1, \dots, X_k \subseteq W$,

$$f_i(X_1, \dots, X_k) = \{w \in W : R_i(w, x_1, \dots, x_k) \text{ for some } x_1 \in X_1, \dots, x_k \in X_k\}.$$

Given a class \mathcal{C} of frames of the same signature, we denote by $\mathbf{Cm } \mathcal{C}$ the class of complex algebras of frames in \mathcal{C} . The starting point of the duality between Kripke complete modal logics and BAOs is the following well-known property. For any class \mathcal{C} of frames, and for any frame \mathfrak{F} of the same signature,

$$\mathfrak{F} \models \text{Log}(\mathcal{C}) \iff \mathbf{Cm } \mathfrak{F} \in \mathbf{HSP } \mathbf{Cm } \mathcal{C}. \quad (1)$$

The following general result shows that if \mathcal{C} satisfies some closure conditions, then

\mathbf{H} is not needed in generating the variety corresponding to $\mathbf{Log}(\mathcal{C})$:

Theorem 2.1 (Goldblatt [11]) *If \mathcal{C} is a class of frames that is closed under ultraproducts, then $\mathbf{SP Cm Gsf C}$ is a canonical variety.*

Let us have a closer look at the subdirectly irreducible algebras of these varieties.

Lemma 2.2 *For any class \mathcal{C} of frames, the subdirectly irreducible members of $\mathbf{SP Cm Gsf C}$ belong to $\mathbf{S Cm Gsf}_p \mathcal{C}$.*

Proof. Let $\mathfrak{A} \in \mathbf{SP Cm Gsf C}$ and let $\mathfrak{A} \hookrightarrow \prod_{i \in I} \mathfrak{A}_i$ be a subdirect embedding, for some $\mathfrak{A}_i \in \mathbf{S Cm Gsf C}$, $i \in I$. If \mathfrak{A} is subdirectly irreducible then there is an $i \in I$ such that \mathfrak{A} is isomorphic to \mathfrak{A}_i , and so \mathfrak{A} is isomorphic to a subalgebra of $\mathbf{Cm } \mathfrak{F}$ for some $\mathfrak{F} \in \mathbf{Gsf C}$. Then for each point x in \mathfrak{F} , $\mathfrak{F}^x \in \mathbf{Gsf}_p \mathbf{Gsf C} \subseteq \mathbf{Gsf}_p \mathcal{C}$. It is not hard to show (see e.g. [10, 3.3]) that $\mathbf{Cm } \mathfrak{F} \hookrightarrow \prod_{x \in \mathfrak{F}} \mathbf{Cm } \mathfrak{F}^x$ is a (subdirect) embedding. So there exist subalgebras \mathfrak{B}_x of $\mathbf{Cm } \mathfrak{F}^x$ such that $\mathfrak{A} \hookrightarrow \prod_{x \in \mathfrak{F}} \mathfrak{B}_x$ is a subdirect embedding as well. As \mathfrak{A} is subdirectly irreducible, there is some x in \mathfrak{F} such that \mathfrak{A} is isomorphic to \mathfrak{B}_x , and so $\mathfrak{A} \in \mathbf{S Cm Gsf}_p \mathcal{C}$. \square

Now Theorem 2.1 and Lemma 2.2 imply the following characterisation of varieties generated by certain classes of complex algebras.

Theorem 2.3 *If \mathcal{C} is a class of frames that is closed under ultraproducts and point-generated subframes, then $\mathbf{SP Cm C} = \mathbf{HSP Cm C}$ is a canonical variety.*

We can also have a ‘dual’ structural characterisation of subdirectly irreducible algebras of these varieties. Recall that an *ultrafilter* of a BAO $\mathfrak{A} = (A, \wedge, -, f_i)_{i \in I}$ is any subset μ of A such that, for all $a, b \in A$,

- if $a \in \mu$ and $a \wedge b = a$ then $b \in \mu$;
- if $a, b \in \mu$ then $a \wedge b \in \mu$;
- $a \in \mu$ iff $-a \notin \mu$.

Let $Uf(A)$ denote the set of all such ultrafilters. Given a BAO $\mathfrak{A} = (A, \wedge, -, f_i)_{i \in I}$, we denote by $\mathfrak{Uf } \mathfrak{A} = (Uf(A), R_i)_{i \in I}$ its *ultrafilter frame*, where for each k -ary function f_i , R_i is the following $k + 1$ -ary relation: for any $\mu, \nu_1, \dots, \nu_k \in Uf(A)$,

$$R_i(\mu, \nu_1, \dots, \nu_k) \quad \text{iff} \quad \forall a_1 \in \nu_1, \dots, a_k \in \nu_k \quad f_i(a_1, \dots, a_k) \in \mu.$$

The *ultrafilter extension* of a frame \mathfrak{F} is $\mathfrak{Ue } \mathfrak{F} = \mathfrak{Uf } \mathbf{Cm } \mathfrak{F}$.

Theorem 2.4 *Let \mathcal{C} be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every subdirectly irreducible algebra \mathfrak{A} ,*

$$\mathfrak{A} \in \mathbf{SP Cm C} \iff \mathfrak{A} \in \mathbf{S Cm C} \iff \mathfrak{Uf } \mathfrak{A} \text{ is a } p\text{-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Proof. \Leftarrow : By Jónsson and Tarski’s [19] theorem, \mathfrak{A} is embeddable into $\mathbf{Cm } \mathfrak{Uf } \mathfrak{A}$. And by duality, $\mathbf{Cm } \mathfrak{Uf } \mathfrak{A}$ is embeddable into $\mathbf{Cm } \mathfrak{G} \in \mathbf{Cm C}$.

\Rightarrow : If $\mathfrak{A} \in \mathbf{SP Cm C}$ then there is a subdirect embedding $\mathfrak{A} \hookrightarrow \prod_{i \in I} \mathfrak{A}_i$, for some $\mathfrak{A}_i \in \mathbf{S Cm C}$, $i \in I$. As \mathfrak{A} is subdirectly irreducible, there is an $i \in I$ such that \mathfrak{A} is isomorphic to \mathfrak{A}_i , that is, \mathfrak{A} is isomorphic to a subalgebra of $\mathbf{Cm } \mathfrak{F}$ for some $\mathfrak{F} \in \mathcal{C}$.

By duality, $\mathcal{Uf}\mathcal{A}$ is a p-morphic image of $\mathcal{Ue}\mathcal{F}$. As $\mathcal{Ue}\mathcal{F}$ is a p-morphic image of an ultrapower of \mathcal{F} (see [7,1,2]) and \mathcal{C} is closed under taking ultraproducts, the proof is completed. \square

As a consequence, we obtain a characterisation of “non-standard” frames for certain logics of the form $\text{Log}(\mathcal{C})$:

Corollary 2.5 *Let \mathcal{C} be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every rooted frame \mathcal{F} ,*

$$\mathcal{F} \models \text{Log}(\mathcal{C}) \quad \iff \quad \mathcal{Ue}\mathcal{F} \text{ is a p-morphic image of some } \mathcal{G} \in \mathcal{C}.$$

Proof. By (1) and Theorem 2.3,

$$\mathcal{F} \models \text{Log}(\mathcal{C}) \quad \iff \quad \text{Cm}\mathcal{F} \in \mathbf{SP Cm}\mathcal{C}.$$

As the complex algebra of a rooted frame is subdirectly irreducible [10], the statement follows from Theorem 2.4. \square

As the ultrafilter extension of a finite frame is isomorphic to the frame itself, we obtain:

Corollary 2.6 *Let \mathcal{C} be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every finite rooted frame \mathcal{F} ,*

$$\mathcal{F} \models \text{Log}(\mathcal{C}) \quad \iff \quad \mathcal{F} \text{ is a p-morphic image of some } \mathcal{G} \in \mathcal{C}.$$

Now we would like to apply these general results to various classes of n -dimensional product frames, whenever $0 < n < \omega$. To this end, observe that the product operation commutes with ultraproducts and point-generated subframes:

Claim 2.7 *Let U be an ultrafilter over some index set I , and let \mathfrak{F}_k^i be a 1-frame, for $i \in I, k < n$. Then:*

$$\prod_{i \in I} (\mathfrak{F}_0^i \times \cdots \times \mathfrak{F}_{n-1}^i) / U \quad \text{is isomorphic to} \quad \left(\prod_{i \in I} \mathfrak{F}_0^i / U \right) \times \cdots \times \left(\prod_{i \in I} \mathfrak{F}_{n-1}^i / U \right).$$

Claim 2.8 *Let $\mathfrak{F} = \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1}$ and \mathbf{x} be a point in \mathfrak{F} . Then:*

$$\mathfrak{F}^{\mathbf{x}} = \mathfrak{F}_0^{x_0} \times \cdots \times \mathfrak{F}_{n-1}^{x_{n-1}}.$$

Given classes \mathcal{C}_i of 1-frames, for $i < n$, let us define

$$\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1} = \{ \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} : \mathfrak{F}_i \in \mathcal{C}_i, i < n \}.$$

As a consequence of Claims 2.7 and 2.8, we obtain:

Proposition 2.9 *If, for $i < n$, \mathcal{C}_i is a class of 1-frames that is closed under ultraproducts and point-generated subframes, then the class $\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ of n -dimensional product frames is closed under ultraproducts and point-generated subframes.*

Now, by Theorem 2.3, (1) and Corollary 2.5, we have:

Theorem 2.10 *If, for $i < n$, \mathcal{C}_i is a class of 1-frames that is closed under ultra-products and point-generated subframes, then:*

- (i) $\mathbf{SP Cm}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}) = \mathbf{HSP Cm}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1})$ is a canonical variety.
- (ii) $\mathbf{Log}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1})$ is a canonical n -modal logic.
- (iii) For every rooted n -frame \mathfrak{F} ,

$$\mathfrak{F} \models \mathbf{Log}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}) \iff \mathfrak{Ue} \mathfrak{F} \text{ is a } p\text{-morphic image of some } \mathfrak{G} \in \mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}.$$

Remark 2.11 The condition of Theorem 2.10 clearly holds if each \mathcal{C}_i is defined by a set of 1-modal formulas having first-order correspondents, such as the classes of all frames of well-known modal logics like **K**, **K4**, **K4.3**, **S4.3**, **S5**, $\mathbf{Log}\{(\mathbb{Q}, <)\}$.

In particular, the classes \mathcal{C}_{all}^n and \mathcal{C}_{equiv}^n introduced in Section 1 are examples of classes of the form $\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ within the scope of Theorem 2.10. So, for every rooted n -frame \mathfrak{F} ,

$$\mathfrak{F} \models \mathbf{K}^n \iff \mathfrak{Ue} \mathfrak{F} \text{ is a } p\text{-morphic image of some } \mathfrak{G} \in \mathcal{C}_{all}^n, \quad (2)$$

$$\mathfrak{F} \models \mathbf{S5}^n \iff \mathfrak{Ue} \mathfrak{F} \text{ is a } p\text{-morphic image of some } \mathfrak{G} \in \mathcal{C}_{equiv}^n. \quad (3)$$

Also, $\mathbf{SP Cm} \mathcal{C}_{all}^n$ and $\mathbf{SP Cm} \mathcal{C}_{equiv}^n$ are canonical varieties. The latter is a variety well-known in algebraic logic: the variety of n -dimensional representable diagonal-free cylindric algebras [14].

The following lemma shows that any n -frame having n equivalence relations and being a p -morphic image of an arbitrary n -dimensional product frame is also a p -morphic image of a product of n equivalence frames.

Lemma 2.12 *Let $n > 0$ be an arbitrary natural number, and let $\mathfrak{F} = (W, T_i)_{i < n}$ be an n -frame such that every T_i is an equivalence relation, for $i < n$. Suppose that $f : \mathfrak{G}_0 \times \cdots \times \mathfrak{G}_{n-1} \rightarrow \mathfrak{F}$ is a surjective p -morphism, for some 1-frames $\mathfrak{G}_i = (U_i, R_i)$, $i < n$. Then there exist 1-frames $\mathfrak{G}_i^* = (U_i, R_i^*)$, $i < n$, such that*

- each R_i^* is an equivalence relation extending R_i , and
- $f : \mathfrak{G}_0^* \times \cdots \times \mathfrak{G}_{n-1}^* \rightarrow \mathfrak{F}$ is still a surjective p -morphism.

Proof. In order to obtain the ‘equivalence-closure’ R_i^* of each R_i , one can add the missing pairs step by step, like it is done for the $n = 2$ case in the proof of [8, Lemma 5.8]. The fact that now n is an arbitrary natural number does not make any difference. \square

Remark 2.13 Note that a similar proof would prove a stronger statement. The property of each T_i being an equivalence relation can be replaced with any property of T_i that can be defined by a set of *universal Horn* formulas in the first-order language having a binary predicate symbol and equality (and there can be different such properties for different i).

As a consequence of Theorem 2.10 and Lemma 2.12 we obtain:

Theorem 2.14 *Let L be any canonical n -modal logic with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then $\mathbf{S5}^n$ is finitely axiomatisable over L : $\mathbf{S5}^n$ is the smallest n -modal logic containing L and the **S5**-axioms for \diamond_i , $i < n$.*

Proof. One inclusion is clear, let us prove the other. The **S5**-axioms are well-known examples of Sahlqvist formulas, and their first-order correspondent is the property of being an equivalence relation. So, by Sahlqvist's completeness theorem, the smallest n -modal logic containing L and the **S5**-axioms for \diamond_i , $i < n$ is canonical, and so Kripke complete. So it is enough to show that every rooted n -frame \mathfrak{F} for this logic is a frame for **S5** ^{n} .

Take such an n -frame \mathfrak{F} . As \mathfrak{F} is a frame for $\mathbf{K}^n = \text{Log}(\mathcal{C}_{all}^n)$, by (2), $\mathcal{Ue}\mathfrak{F}$ is a p-morphic image of some n -dimensional product frame \mathfrak{G} . As \mathfrak{F} validates the canonical **S5**-axioms, they also hold in $\mathcal{Ue}\mathfrak{F}$, and so all the relations in $\mathcal{Ue}\mathfrak{F}$ are equivalence relations. Now by Lemma 2.12, $\mathcal{Ue}\mathfrak{F}$ is a p-morphic image of some $\mathfrak{G}^* \in \mathcal{C}_{equiv}^n$, and so by (3), \mathfrak{F} is a frame for **S5** ^{n} = $\text{Log}(\mathcal{C}_{equiv}^n)$. \square

Remark 2.15 By Remarks 2.11 and 2.13 we can have similar statements for any $\text{Log}(\mathcal{K})$ in place of **S5** ^{n} , whenever $\mathcal{K} = \mathcal{C}_0 \times \dots \times \mathcal{C}_{n-1}$ for some classes \mathcal{C}_i of 1-frames, each of which is definable by Sahlqvist formulas having universal Horn first-order correspondents.

Theorem 2.14 shows that any negative result on the equational axiomatisation of the variety on n -dimensional representable diagonal-free cylindric algebras (such as its non-finiteness [18], for $n \geq 3$) transfers not only to its logic counterpart **S5** ^{n} , but also to other many-dimensional modal logics like \mathbf{K}^n . In other words, this theorem also means that all the complexities of a possible axiomatisation of **S5** ^{n} come from the many-dimensional nature of the product frames and are already present in an axiomatisation of \mathbf{K}^n . Though, by a general result of [9], \mathbf{K}^n is known to be recursively enumerable, an axiomatisation of \mathbf{K}^n should be quite complex, whenever $n \geq 3$: any such axiomatisation should contain modal formulas of arbitrary modal depth for each modality [20], and infinitely many propositional variables [22]. (At the moment we cannot use Theorem 2.14 to infer the latter, as it is not known whether **S5** ^{n} can be axiomatised using finitely many variables, whenever $n \geq 3$.) As Theorem 1.1 above shows, it will be quite hard to find an explicit axiomatisation for \mathbf{K}^n , as any such must contain n -modal formulas having no first-order correspondents.

3 Frames constructed from graphs

This and the next section are devoted to the proof of Theorem 1.1. Throughout, we fix a natural number $n \geq 3$. We will use n as a notation for both this number and for the set $\{0, \dots, n-1\}$. In order to show Theorem 1.1, we will give n -frames \mathfrak{G}_k , for $k < \omega$, such that each \mathfrak{G}_k is a frame for **S5** ^{n} , but any non-principal ultraproduct of the \mathfrak{G}_k s is not a frame for \mathbf{K}^n .

We will use a construction of Hirsch and Hodkinson [15], so let us introduce the necessary notions. To begin with, let us enrich n -frames by adding some unary relations. An $n\delta$ -frame is a relational structure of the form $\mathfrak{F} = (W, T_i, E_{ij})_{i,j < n}$ where $(W, T_i)_{i < n}$ is an n -frame and $E_{ij} \subseteq W$ for all $i, j < n$. For any n -dimensional product frame $\mathfrak{F} = (W_0 \times \dots \times W_{n-1}, \bar{R}_i)_{i < n}$, we define an $n\delta$ -frame \mathfrak{F}^δ by taking

$$\mathfrak{F}^\delta = (W_0 \times \dots \times W_{n-1}, \bar{R}_i, \delta_{ij})_{i,j < n},$$

where $\delta_{ij} = \{\mathbf{w} \in W_0 \times \cdots \times W_{n-1} : w_i = w_j\}$, for $i, j < n$. These δ_{ij} s are called *diagonal elements*. Now let

$$\mathcal{C}_{cube}^{n\delta} = \{(\underbrace{\mathfrak{F} \times \cdots \times \mathfrak{F}}_n)^\delta : \mathfrak{F} = (U, U \times U) \text{ for some non-empty set } U\}.$$

Note that if $\mathfrak{F}^\delta \in \mathcal{C}_{cube}^{n\delta}$ then $\mathfrak{F} \in \mathcal{C}_{equiv}^n$. Using Claims 2.7 and 2.8, it is not hard to see that $\mathcal{C}_{cube}^{n\delta}$ is closed under ultraproducts and point-generated subframes. So, by Theorem 2.3, $\mathbf{SP Cm} \mathcal{C}_{cube}^{n\delta}$ is a canonical variety, well-known in algebraic logic: the variety of *n-dimensional representable cylindric algebras* [14].

Next, we define special $n\delta$ -frames with the help of graphs. By a *graph* we mean a pair (Γ, E) , where Γ is non-empty set and E is an irreflexive and symmetric binary relation on Γ (the *edges*). We identify a graph with its underlying set Γ of *nodes*. Given a graph $\Gamma = (\Gamma, E)$, a set $X \subseteq \Gamma$ is called *independent*, if $(x, y) \notin E$ whenever $x, y \in X$. The *chromatic number* $\chi(\Gamma)$ of Γ is the smallest $k < \omega$ such that Γ can be partitioned into k independent sets, and ∞ is there is no such k . An *ultrafilter on Γ* is an ultrafilter of the Boolean algebra of all subsets of Γ . For any graph Γ and $n < \omega$, we define the graph $\Gamma \times n$ as n disjoint copies of Γ , with all possible edges between distinct copies being added. For notions not defined here and general information on graphs, see [5].

Given a graph Γ , Hirsch and Hodkinson [15] define an $n\delta$ -frame

$$\mathfrak{F}_\Gamma = (H_\Gamma, \equiv_i, D_{ij})_{i, j < n}$$

as follows.

- H_Γ is the set of all pairs (K, \sim) , where $K : n \rightarrow \Gamma \times n$ is a partial map, and \sim is an equivalence relation on n , satisfying one of the following properties:
 - Either: all distinct $i, j < n$ are not \sim -equivalent, $K(i)$ is defined for all $i < n$, and $\{K(0), \dots, K(n-1)\}$ is not an independent set in $\Gamma \times n$.
 - Or: $\{i, j\}$ is a 2-element \sim -class, all other \sim -classes are singletons, $K(i)$ and $K(j)$ are both defined and $K(i) = K(j)$, and $K(k)$ is not defined for $k \neq i, j$.
 - Or: the number of \sim -classes is $\leq n - 2$ and $K = \emptyset$.
- For every $i < n$, \equiv_i is a binary relation on H_Γ defined by

$$(K, \sim) \equiv_i (K', \sim') \quad \text{iff} \quad \sim \upharpoonright_{n-\{i\}} = \sim' \upharpoonright_{n-\{i\}}, \text{ and} \\ \text{either both } K(i) \text{ and } K'(i) \text{ are undefined,} \\ \text{or both } K(i) \text{ and } K'(i) \text{ are defined and } K(i) = K'(i).$$

- For all $i, j < n$, D_{ij} is the following subset of H_Γ :

$$D_{ij} = \{(K, \sim) : i \sim j\}.$$

The following two propositions are proved in [15]:

Proposition 3.1 [15, Prop.5.2]

If $\chi(\Gamma) = \infty$ then $\mathbf{Cm} \mathfrak{F}_\Gamma$ is an n-dimensional representable cylindric algebra.

Proposition 3.2 [15, Prop.5.4]

If Γ is infinite and $\chi(\Gamma) < \omega$, then $\mathfrak{Cm} \mathfrak{F}_\Gamma$ is not an n -dimensional representable cylindric algebra.

Observe that $\mathfrak{Cm} \mathfrak{F}_\Gamma$ is a BAO of the form $(A, \wedge, -, c_i, d_{ij})_{i,j < n}$, where each c_i is a unary function on A and each d_{ij} is an element of A . If we forget about the d_{ij} s, we obtain what is called the *diagonal-free reduct* of $\mathfrak{Cm} \mathfrak{F}_\Gamma$. It should be clear that this diagonal-free reduct is in fact $\mathfrak{Cm} \mathfrak{F}_\Gamma^-$, where \mathfrak{F}_Γ^- is the n -frame $(H_\Gamma, \equiv_i)_{i < n}$.

We would like to have the diagonal-free “analogues” of Propositions 3.1 and 3.2. On the one hand, it is straightforward to see that if $\mathfrak{Cm} \mathfrak{F}_\Gamma$ is an n -dimensional representable cylindric algebra, that is, it belongs to $\mathbf{SPCm} \mathcal{C}_{cube}^{n\delta}$, then its diagonal-free reduct $\mathfrak{Cm} \mathfrak{F}_\Gamma^-$ belongs to $\mathbf{SPCm} \mathcal{C}_{equiv}^n$. So by (1) and Prop. 3.1 we obtain:

Proposition 3.3 If $\chi(\Gamma) = \infty$ then \mathfrak{F}_Γ^- is a frame for $\mathbf{S5}^n$.

On the other hand, having the analogue of Prop. 3.2 is not so easy. As is well-known in algebraic logic, there are $n\delta$ -frames \mathfrak{G} such that though $\mathfrak{Cm} \mathfrak{G}$ is *not* an n -dimensional representable cylindric algebra, yet its diagonal-free reduct $\mathfrak{Cm} \mathfrak{G}^-$ is an n -dimensional representable diagonal-free cylindric algebra [14]. We will show that if Γ is infinite and $\chi(\Gamma) < \infty$ then for $\mathfrak{G} = \mathfrak{F}_\Gamma$ this is not the case: $\mathfrak{Cm} \mathfrak{F}_\Gamma^-$ is not an n -dimensional representable diagonal-free cylindric algebra, and so \mathfrak{F}_Γ^- is not a frame for $\mathbf{S5}^n$.

Let us begin with showing some further properties of \mathfrak{F}_Γ :

Claim 3.4 (i) For every $i < n$, \equiv_i is an equivalence relation, and $D_{ii} = H_\Gamma$.

(ii) For all $i, j < n$, \equiv_i and \equiv_j commute.

(iii) For all $i, j, k < n$, $i \neq j$, $k \neq i, j$ and for all $(K \sim) \in H_\Gamma$,

$$(K, \sim) \in D_{ij} \quad \text{iff} \quad \text{there is } (K', \sim') \in D_{ik} \cap D_{kj} \text{ such that } (K, \sim) \equiv_k (K', \sim').$$

(iv) For all $i, j < n$, $i \neq j$, if $(K, \sim), (K', \sim') \in D_{ij}$ and $(K, \sim) \equiv_i (K', \sim')$, then $(K, \sim) = (K', \sim')$.

(v) \mathfrak{F}_Γ is rooted.

Proof. The proofs of items (i) and (ii) are tiresome at places, but straightforward.

(iii): Fix some $k \neq i, j$. First, let $(K, \sim) \in D_{ij}$. Then $i \sim j$ and $K(k)$ is not defined for $k \neq i, j$. Let $K' = \emptyset$ and \sim' such that $\sim'|_{n-\{k\}} = \sim|_{n-\{k\}}$ and $k \sim' i \sim' j$. Then $(K', \sim') \in H_\Gamma$ as required. For the other direction, let $(K', \sim') \in D_{ik} \cap D_{kj}$ and $(K, \sim) \equiv_k (K', \sim')$. Then $i \sim' k \sim' j$ and $\sim'|_{n-\{k\}} = \sim|_{n-\{k\}}$, so $i \sim j$, thus $(K, \sim) \in D_{ij}$.

(iv): If $(K, \sim), (K', \sim') \in D_{ij}$ and $(K, \sim) \equiv_i (K', \sim')$, then $i \sim j$, $i \sim' j$ and $\sim|_{n-\{i\}} = \sim'|_{n-\{i\}}$. Therefor $\sim = \sim'$ follows. Then there are two cases: either all of $K(i)$, $K(j)$, $K'(i)$, $K'(j)$ are defined and equal, or none of them is defined. In either case, $K = K'$ follows.

(v): (cf. [15, proof of Lemma 5.1]) We show that $(\emptyset, n \times n) \in H_\Gamma$ is suitable as root. To this end, take any $(K, \sim) \in H_\Gamma$. For any $i < n$, define a partial function

$K_i : n \rightarrow \Gamma \times n$ by taking

$$K_i(j) = \begin{cases} K(i), & \text{if } j = 0 \text{ or } j = i, \text{ and } K(i) \text{ is defined,} \\ \text{undefined,} & \text{else.} \end{cases}$$

Let \sim_i be the unique equivalence relation such that $\sim_i|_{n-\{i\}} = \sim|_{n-\{i\}}$ and $i \sim_i 0$. Then $(K_i, \sim_i) \in H_\Gamma$ and $(K, \sim) \equiv_i (K_i, \sim_i)$. So we have

$$(K, \sim) \equiv_1 (K_1, \sim_1) \equiv_2 (K_{12}, \sim_{12}) \cdots \equiv_{n-1} (K_{12\dots n-1}, \sim_{12\dots n-1}).$$

As $n \geq 3$, we have $0 \sim_{12} 1 \sim_{12} 2$, so $K_{12} = \cdots = K_{12\dots n-1} = \emptyset$. Also, $\sim_{12\dots n-1} = n \times n$. Therefore, by item (i), $(\emptyset, n \times n)$ is a root of \mathfrak{F}_Γ . \square

Properties (i)–(iv) above form the definition of what is called in algebraic logic an *n-dimensional cylindric atom structure* (see [13, 2.7.40]). Complex algebras of these special $n\delta$ -frames belong to the variety of *n-dimensional cylindric algebras*. The interested reader can find the definition of this class in e.g. [13]. Here we only use that, being a variety, the class of *n-dimensional cylindric algebras* is closed under subalgebras. So, in particular, by Claim 3.4 we have that

$$\text{any subalgebra of } \mathfrak{Cm} \mathfrak{F}_\Gamma \text{ is an } n\text{-dimensional cylindric algebra.} \quad (4)$$

An element a in an algebra $\mathfrak{A} = (A, \wedge, -, c_i, d_{ij})_{i,j < n}$ is called *<n-dimensional*, if there is some $i < n$ such that $c_i(a) = a$. We will use the following result:

Theorem 3.5 (Johnson [18], see also [12,14])

Let \mathfrak{A} be an n-dimensional cylindric algebra that is generated by its <n-dimensional elements. If the diagonal-free reduct \mathfrak{A}^- of \mathfrak{A} is an n-dimensional representable diagonal-free cylindric algebra, then \mathfrak{A} is an n-dimensional representable cylindric algebra.

In Section 4 below we will define a subalgebra \mathfrak{A}_Γ of $\mathfrak{Cm} \mathfrak{F}_\Gamma$ and show the following two statements:

Proposition 3.6 \mathfrak{A}_Γ is an n-dimensional cylindric algebra generated by its <n-dimensional elements.

Proposition 3.7 (cf. [15, Prop.5.4])

If Γ is infinite and $\chi(\Gamma) < \omega$, then \mathfrak{A}_Γ is not an n-dimensional representable cylindric algebra.

Now if Γ is infinite and $\chi(\Gamma) < \omega$ then, by Theorem 3.5, the diagonal-free reduct \mathfrak{A}_Γ^- of \mathfrak{A}_Γ is not an *n-dimensional representable diagonal-free cylindric algebra*, that is, it does not belong to $\mathbf{SP Cm} \mathcal{C}_{equiv}^n$. As \mathfrak{A}_Γ^- is a subalgebra of $\mathfrak{Cm} \mathfrak{F}_\Gamma^-$, it follows that $\mathfrak{Cm} \mathfrak{F}_\Gamma^-$ does not belong to $\mathbf{SP Cm} \mathcal{C}_{equiv}^n$ either. So, by Claim 3.4(v) and (3), $\mathfrak{Ue} \mathfrak{F}_\Gamma^-$ is not a p-morphic image of a product of *n* equivalence frames. On the other hand, by Claim 3.4(i), all the relations \equiv_i in \mathfrak{F}_Γ^- are equivalence relations, for $i < n$. Therefore, the *n*-frame \mathfrak{F}_Γ^- validates the canonical **S5**-axioms, for all $i < n$, so they also hold in $\mathfrak{Ue} \mathfrak{F}_\Gamma^-$, meaning that all its relations are equivalence relations as well.

So, by Lemma 2.12, $\mathfrak{Ue} \mathfrak{F}_\Gamma^-$ is not a p-morphic image of *any* product frame. So, by (2), we have the required analogue of Prop. 3.2:

Proposition 3.8 *If Γ is infinite and $\chi(\Gamma) < \omega$, then \mathfrak{F}_Γ^- is not a frame for \mathbf{K}^n .*

Now we can complete the proof of Theorem 1.1 precisely as it is done in the proof of [15, Thm.6.1]: It is not hard to see that if U is a non-principal ultrafilter over some index set I , then

$$\prod_{i \in I} \mathfrak{F}_{\Gamma_i}^- / U \text{ is isomorphic to } \mathfrak{F}_{\prod_{i \in I} \Gamma_i / U}^-. \quad (5)$$

So what is left is to have a sequence $(\Gamma_k)_{k < \omega}$ of graphs such that

- $\chi(\Gamma_k) = \infty$ for all $k < \omega$.
- If Γ is any non-principal ultraproduct of the Γ_k , then Γ is infinite and $\chi(\Gamma) < \omega$.

As is shown in [15], one can have such a sequence of graphs by using Erdős's famous theorem [6]. Now let L be any set of n -modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then, by Prop. 3.3, each $\mathfrak{F}_{\Gamma_k}^-$ is a frame for L . On the other hand, by (5) and Prop. 3.8, any non-principal ultraproduct of the $\mathfrak{F}_{\Gamma_k}^-$ is not a frame for \mathbf{K}^n , and so not a frame for L .

4 The algebra \mathfrak{A}_Γ

What is left is to define a subalgebra \mathfrak{A}_Γ of $\mathfrak{Cm} \mathfrak{F}_\Gamma$, and prove Propositions 3.6 and 3.7 about it. We define \mathfrak{A}_Γ using notions introduced in [15, Defs. 4.1, 4.4]. To this end, for $i < n$, let

$$F_i = \bigcap_{j, k \neq i, j \neq k} (H_\Gamma - D_{jk}) = \{(K, \sim) \in H_\Gamma : K(i) \text{ is defined}\}.$$

Now, for any $X \subseteq \Gamma \times n$, put

$$X^{(i)} = \{(K, \sim) \in F_i : K(i) \in X\},$$

and let \mathfrak{A}_Γ be the subalgebra of $\mathfrak{Cm} \mathfrak{F}_\Gamma$ generated by the set

$$\{X^{(i)} : i < n, X \subseteq \Gamma \times n\}.$$

Proof of Prop. 3.6. By (4), \mathfrak{A}_Γ is an n -dimensional cylindric algebra. Now take any $i < n$ and $X \subseteq \Gamma \times n$. Let $(K, \sim) \in X^{(i)}$ and $(K', \sim') \in H_\Gamma$ such that $(K, \sim) \equiv_i (K', \sim')$. Then both $K(i)$ and $K'(i)$ are defined, $(K', \sim') \in F_i$ and $K(i) = K'(i)$, so $(K', \sim') \in X^{(i)}$ as well. This shows that $c_i(X^{(i)}) = X^{(i)}$, so $X^{(i)}$ is $< n$ -dimensional.

Proof of Prop. 3.7. We establish a connection between ultrafilters of \mathfrak{A}_Γ and ultrafilters over $\Gamma \times n$, just like it is done in [15] between ultrafilters of $\mathfrak{Cm} \mathfrak{F}_\Gamma$ and ultrafilters over $\Gamma \times n$.

For any $i < n$, let E_i denote the binary relation corresponding to c_i in the ultrafilter frame of \mathfrak{A}_Γ . For any $S \subseteq F_i$, put $S(i) = \{K(i) : (K, \sim) \in S\}$. For any $i < n$, and any ultrafilter μ of \mathfrak{A}_Γ , let

$$\mu(i) = \{S(i) : S \in \mu, S \subseteq F_i\}.$$

Claim 4.1 (analogue of [15, Lemma 4.6])

Let μ be an ultrafilter of \mathfrak{A}_Γ such that $F_i \in \mu$ for some $i < n$. Then:

- (i) $\mu(i)$ is an ultrafilter on $\Gamma \times n$.
- (ii) If $j < n$ and $D_{ij} \in \mu$, then $F_j \in \mu$ and $\mu(j) = \mu(i)$.
- (iii) For any ultrafilter ν of \mathfrak{A}_Γ , we have $\mu E_i \nu$ iff $F_i \in \nu$ and $\mu(i) = \nu(i)$.

Proof. (i): An arbitrary element of $\mu(i)$ is of the form $S(i)$ for some $S \in \mu$, $S \subseteq F_i$. Suppose that $S(i) \subseteq X \subseteq \Gamma \times n$. Then it is not hard to see that $S \subseteq S(i)^{(i)} \subseteq X^{(i)}$. As $X^{(i)}$ is an element of \mathfrak{A}_Γ and μ is an ultrafilter of \mathfrak{A}_Γ , $X^{(i)} \in \mu$ follows. We also have $X^{(i)} \subseteq F_i$. So $X = X^{(i)}(i) \in \mu(i)$.

The proofs of the other two ultrafilter-properties, and of (ii) and (iii) are the same as those of the corresponding items in [15, Lemma 4.6]. \square

Now we can complete the proof of Prop. 3.7 by following precisely the same steps as in the proof of [15, Prop.5.4]), using ultrafilters of \mathfrak{A}_Γ in place of ultrafilters of $\mathfrak{Cm} \mathfrak{F}_\Gamma$. If $\chi(\Gamma) < \omega$, then also $\chi(\Gamma \times n) < \omega$. So $\Gamma \times n = I_0 \cup \dots \cup I_{k-1}$ for some natural number k and independent sets I_j , for $j < k$. So, for every ultrafilter μ on $\Gamma \times n$, there is a unique $j < k$ such that $I_j \in \mu$. As Γ is infinite, so is H_Γ , and so is \mathfrak{A}_Γ .

Now suppose that \mathfrak{A}_Γ is an n -dimensional representable cylindric algebra. As is shown in [15, Lemma 5.1], every subalgebra of $\mathfrak{Cm} \mathfrak{F}_\Gamma$ is subdirectly irreducible, therefore so is \mathfrak{A}_Γ . Thus, by Theorem 2.4, $\mathfrak{Uf} \mathfrak{A}_\Gamma$ is a p-morphic image of some frame from $\mathcal{C}_{cube}^{n\delta}$, that is, there exist an infinite set U and a surjective function $h : U^n \rightarrow \{\text{ultrafilters of } \mathfrak{A}_\Gamma\}$ such that

- (h1) for all $i < n$, $\mathbf{a}, \mathbf{b} \in U^n$, if $a_j = b_j$ for all $j < n$, $j \neq i$, then $h(\mathbf{a}) E_i h(\mathbf{b})$,
- (h2) for all $i, j < n$, $\mathbf{a} \in U^n$, $a_i = a_j$ iff $D_{ij} \in h(\mathbf{a})$.

(We will not use the ‘backward’ condition w.r.t. E_i .) So if $\mathbf{a} \in U^n$ is such that all the a_i are different for $i < n$ then, by (h2) and Claim 4.1(i),

$$(h(\mathbf{a})(0), \dots, h(\mathbf{a})(n-1))$$

is an n -tuple of $(\Gamma \times n)$ -ultrafilters. We show that for each $i < n$, $h(\mathbf{a})(i)$ depends only on the set $\{a_0, \dots, a_{n-1}\} - \{a_i\}$. That is, such a function h determines of what is called in [15] a *patch system*.

Claim 4.2 Let $i, j < n$ and $\mathbf{a}, \mathbf{b} \in U^n$ be such that

- $a_k \neq a_\ell$ whenever $k, \ell \neq i$, $k, \ell < n$,
- $b_k \neq b_\ell$ whenever $k, \ell \neq j$, $k, \ell < n$, and
- $\{a_k : k < n, k \neq i\} = \{b_k : k < n, k \neq j\}$.

Then $h(\mathbf{a})(i) = h(\mathbf{b})(j)$.

Proof. This claim is claimed and proved in the proof of [15, Lemma 4.12(2)]. Using ultrafilters of \mathfrak{A}_Γ instead of ultrafilters of $\mathfrak{Cm} \mathfrak{F}_\Gamma$ does not make any difference. \square

As a consequence we obtain:

Claim 4.3 (cf. [15, Def. 4.11, Lemma 4.12(2)])

Given h as above, define a function

$$\partial h : \{n - 1\text{-element subsets of } U\} \rightarrow \{\text{ultrafilters on } \Gamma \times n\}$$

by taking, for every n -element subset A of U an n -tuple $\mathbf{a} \in U^n$ such that $A = \{a_0, \dots, a_{n-1}\} - \{a_i\}$ for some $i < n$ and putting

$$\partial h(A) = h(\mathbf{a})(i).$$

Then ∂h is well-defined.

Take the functions h and ∂h as defined above. As \mathfrak{A}_Γ is infinite, the domain U^n of h should also be infinite. Choose an infinite sequence a_0, a_1, \dots of distinct elements from U , and define a function

$$f : \{n - 1\text{-element subsets of } \omega\} \rightarrow k$$

by taking

$$f(\{i_1, \dots, i_{n-1}\}) = j \quad \text{iff} \quad I_j \in h(\{a_{i_1}, \dots, a_{i_{n-1}}\}).$$

By Ramsey's theorem [23], we may assume that the value of f is constant, say, c . Let $A = \{a_0, \dots, a_{n-1}\}$ and $\mathbf{a} = (a_0, \dots, a_{n-1})$. Then $I_c \in \partial h(A - \{a_i\}) = h(\mathbf{a})(i)$, for each $i < n$. So for every $i < n$ there exists some $S_i \in h(\mathbf{a})$ such that $S_i \subseteq F_i$ and $S_i(i) = I_c$. As $h(\mathbf{a})$ is an ultrafilter of \mathfrak{A}_Γ and $\bigcap_{i < n} S_i \in h(\mathbf{a})$, we have that $\bigcap_{i < n} S_i \neq \emptyset$. Take any $(K, \sim) \in \bigcap_{i < n} S_i$. Then on the one hand, $K(i)$ is defined for all $i < n$, so the set $\{K(0), \dots, K(n-1)\}$ is not independent. (This argument is written in the proof of [15, Lemma 4.10].) On the other hand, as $S_i(i) = I_c$, we have $\{K(0), \dots, K(n-1)\} \subseteq I_c$, so it is independent, a contradiction, completing the proof of Prop. 3.7.

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References

- [1] Benthem, J., *Canonical modal logics and ultrafilter extensions*, Journal of Symbolic Logic **44** (1979), pp. 1–8.
- [2] Benthem, J., *Some kinds of modal completeness*, Studia Logica **39** (1980), pp. 125–141.
- [3] Blackburn, P., M. de Rijke and Y. Venema, “Modal Logic,” Cambridge University Press, 2001.
- [4] Chagrov, A. and M. Zakharyashev, “Modal Logic,” Oxford Logic Guides **35**, Clarendon Press, Oxford, 1997.
- [5] Diestel, R., “Graph Theory,” Graduate Texts in Mathematics **173**, Springer-Verlag, 1997.

- [6] Erdős, P., *Graph theory and probability*, Canadian Journal of Mathematics **11** (1959), pp. 34–38.
- [7] Fine, K., *Some connections between elementary and modal logic*, in: S. Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, North-Holland, Amsterdam, 1975 pp. 15–31.
- [8] Gabbay, D., A. Kurucz, F. Wolter and M. Zakharyashev, “Many-Dimensional Modal Logics: Theory and Applications,” *Studies in Logic and the Foundations of Mathematics* **148**, Elsevier, 2003.
- [9] Gabbay, D. and V. Shehtman, *Products of modal logics. Part I*, Journal of the IGPL **6** (1998), pp. 73–146.
- [10] Goldblatt, R., *Varieties of complex algebras*, Annals of Pure and Applied Logic **38** (1989), pp. 173–241.
- [11] Goldblatt, R., *Elementary generation and canonicity for varieties of Boolean algebras with operators*, Algebra Universalis **34** (1995), pp. 551–607.
- [12] Halmos, P., *Algebraic logic, IV*, Transactions of the AMS **86** (1957), pp. 1–27.
- [13] Henkin, L., J. Monk and A. Tarski, “Cylindric Algebras, Part I,” *Studies in Logic and the Foundations of Mathematics* **64**, North-Holland, 1971.
- [14] Henkin, L., J. Monk and A. Tarski, “Cylindric Algebras, Part II,” *Studies in Logic and the Foundations of Mathematics* **115**, North-Holland, 1985.
- [15] Hirsch, R. and I. Hodkinson, *Strongly representable atom structures of cylindric algebras*, Journal of Symbolic Logic **74** (2009), pp. 811–828.
- [16] Hirsch, R., I. Hodkinson and A. Kurucz, *On modal logics between $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$* , Journal of Symbolic Logic **67** (2002), pp. 221–234.
- [17] Hodkinson, I. and Y. Venema, *Canonical varieties with no canonical axiomatisation*, Trans. Amer. Math. Soc. **357** (2005), pp. 4579–4605.
- [18] Johnson, J., *Nonfinitizability of classes of representable polyadic algebras*, Journal of Symbolic Logic **34** (1969), pp. 344–352.
- [19] Jónsson, B. and A. Tarski, *Boolean algebras with operators. I*, American Journal of Mathematics **73** (1951), pp. 891–939.
- [20] Kurucz, A., *On axiomatising products of Kripke frames*, Journal of Symbolic Logic **65** (2000), pp. 923–945.
- [21] Kurucz, A., *Combining modal logics*, in: P. Blackburn, J. van Benthem and F. Wolter, editors, *Handbook of Modal Logic*, Studies in Logic and Practical Reasoning **3**, Elsevier, 2007 pp. 869–924.
- [22] Kurucz, A., *On axiomatising products of Kripke frames, part II*, in: C. Areces and R. Goldblatt, editors, *Advances in Modal Logic, Volume 7*, College Publications, 2008 pp. 219–230.
- [23] Ramsey, F., *On a problem of formal logic*, Proceedings of the London Mathematical Society **30** (1930), pp. 264–286.
- [24] Segerberg, K., *Two-dimensional modal logic*, Journal of Philosophical Logic **2** (1973), pp. 77–96.