Absolute and Asymptotic Bounds for Online Frequency Allocation in Cellular Networks

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Abstract

Given a cellular (mobile telephone) network, whose geographical coverage area is divided into hexagonal cells, phone calls are serviced by assigning frequencies to them so that no two calls emanating from the same or neighboring cells are assigned the same frequency. Assuming an online arrival of calls, the goal is to minimize the span of frequencies used to serve all of the calls.

By first considering χ-colorable networks, which is a generalization of the (3-colorable) cellular networks, we present a $(\chi + 1)/2$-competitive online algorithm. This algorithm, when applied to cellular networks, is effectively a positive solution to the open problem posed in [3]: Does a 2-competitive online algorithm exist for frequency allocation in cellular networks? We further prove a matching lower bound, which shows that our 2-competitive algorithm is optimal.

We discover that an interesting phenomenon occurs for the online frequency allocation problem when the number of calls considered becomes large: previously-derived optimal bounds on competitive ratios no longer hold true. For cellular networks, we show a new asymptotic lower and upper bounds of 1.5 and 1.9126, respectively, which breaks through the optimal bound of 2 shown above.

Keywords: Online algorithms; Competitive analysis; Frequency allocation; Cellular networks; Multicoloring

1 Introduction

Wireless communication based on Frequency Division Multiplexing (FDM) technology is widely used in the area of mobile computing today. In such FDM networks, e.g., a cellular network, a geographic area is divided into small hexagonal regions or *cells*, each containing one base station. Base stations communicate with each other via a high-speed wired network. Calls between any two clients (even within the same cell) must be established through base stations. When a call is initiated, the nearest base station must allocate a frequency from the available spectrum to the call without causing any interference to other calls. Interference, which distorts the radio signals, occurs when the same frequency is assigned to two different calls emanating from cells that are geographically close to each other. To avoid interference, the temptation is

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to use a large spectrum of frequencies. However, frequency spectrum is a scarce and limited resource and thus efficient utilization of the available spectrum is essential for FDM networks.

Problem Formulation. The *frequency allocation problem* has been studied extensively (see the surveys [1,12,14,17,18]). One of the main applications of the problem is on cellular networks. In this paper, we focus on the widely adopted model of cellular networks, that are the hexagon graphs (see Figure 1), where a cell is represented by a hexagon. We study the online version of the frequency allocation problem [3–5,10,11,13,20], where calls arrive at arbitrary cells over the time and they stay in the network. Once a new call arrives, a frequency, which is represented by an integer, has to be assigned to the call immediately. The assigned frequency must not create interference, i.e., the frequency must not be in use by other calls in the cell of the new call or any cells adjacent to the cell of the new call. The objective of the problem is to minimize the span of frequencies assigned, i.e., the difference between the highest and the lowest frequencies assigned.

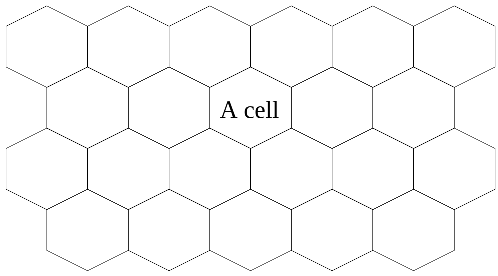

Figure 1: A cellular network modeled as a hexagon graph.

Performance Measures. Competitive analysis [2] is used to measure the performance of our online algorithms. For a sequence σ of calls, let $\mathcal{A}(\sigma)$ denote the cost of the online algorithm $\mathcal{A}$, i.e., the span of frequencies used by the algorithm $\mathcal{A}$, and let $\mathcal{O}(\sigma)$ denote the cost of the optimal off-line algorithm, which knows the whole sequence in advance. The *(absolute) competitive ratio* of algorithm $\mathcal{A}$ is defined to be

$$R_{\mathcal{A}} = \sup_{\sigma} \frac{\mathcal{A}(\sigma)}{\mathcal{O}(\sigma)}.$$

Meanwhile, when the number of calls emanating from each cell is large, the *asymptotic competitive ratio* of algorithm $\mathcal{A}$, which is also a concern in this paper, is defined to be

$$R_{\mathcal{A}}^{\infty} = \limsup_{n\to\infty} \max_{\sigma} \left\{ \frac{\mathcal{A}(\sigma)}{\mathcal{O}(\sigma)} \mid \mathcal{O}(\sigma) = n \right\}.$$

Clearly, for any online algorithm $\mathcal{A}$, $R_{\mathcal{A}}^{\infty} \leq R_{\mathcal{A}}$.

Previous Results. The offline frequency allocation problem on cellular networks is NP-hard [16]. McDiarmid and Reed [16], and Narayanan and Shende [19] both gave a 4/3-approximation algorithm for the problem.

For the online frequency allocation problem, two simple strategies for assigning frequency have been proposed: *fixed allocation algorithm* (FAA) [15] and *greedy algorithm* (Greedy) [3]. FAA colors the hexagon graph with three colors, partitions the frequencies into three disjoint sets, and associates each disjoint set with a distinct color. Each cell is also associated with the set of frequencies with the same color as the cell. When a new call arrives from a cell, the call is assigned the lowest available frequency associated with the cell, so that the frequency does not create interference within the cell. As FAA makes sure that adjacent cells (with different colors) are associated with different frequencies, no interference occurs between adjacent cells. It is easy to see that FAA is 3-competitive when the frequencies are partitioned uniformly into three disjoint sets, e.g., $\{1, 4, 7, \ldots\}, \{2, 5, 8, \ldots\}, \{3, 6, 9, \ldots\}$.

GREEDY assigns the lowest available frequency to a new call so that the frequency does not interfere with calls of the same or adjacent cells. Caragiannis et al. [3] proved that the competitive ratio of GREEDY is at least 17/7 and at most 2.5. Chan et al. [5] gave a tighter analysis to show that GREEDY is 17/7-competitive. Whether there exists a 2-competitive online algorithm was cited as an open problem in [3].

Our Contributions. In this paper we make the following two main contributions.

Firstly, we present a new algorithm, called HYBRID, which is a combination of FAA and GREEDY. The HYBRID algorithm works for the χ-colorable graphs where the coloring is known. Although its idea is simple and its analysis is straightforward, the algorithm solves the open problem posed in [3]. It achieves a competitive ratio of $(\chi + 1)/2$ for the χ-colorable graphs, which implies a 2-competitive algorithm for the 3-colorable cellular networks. Then, we give a matching lower bound to show that HYBRID is indeed optimal for cellular networks.

Secondly, we achieve a better competitive ratio for cellular networks when the number of calls is large. By generalizing the HYBRID algorithm, we are able to guarantee an asymptotic competitive ratio 1.9126, which is better than the absolute competitive ratio 2. In doing so, we show how HYBRID can be adjusted to yield a good asymptotic bound as well as the optimal absolute bound. Finally, we derive a lower bound 1.5 on the asymptotic competitive ratio. Table 1 summaries the results of this paper.

HYBRID	Upper bound	Lower bound
Absolute competitive ratio	2	2
Asymptotic competitive ratio	1.9126	1.5

Table 1: Results of this paper

The rest of the paper is organized as follows. In Section 2, we present the HYBRID algorithm for the frequency allocation problem in χ-colorable networks and show that it is $(\chi + 1)/2$-competitive. As a corollary, HYBRID is thus 2-competitive for cellular networks. We also derive a matching lower bound to show that HYBRID is indeed optimal for such networks. In Section 3, we consider large-scale input and prove the asymptotic upper and lower bounds. Concluding remarks are given in Section 4.

2 Absolute Bounds

In this section the frequency allocation problem is studied on general χ-colorable networks. We introduce the HYBRID algorithm and show that its competitive ratio is at most $(\chi+1)/2$. When applied to the 3-colorable cellular networks, HYBRID becomes 2-competitive.

Having obtained upper bounds for cellular networks, we then study lower bounds. We prove that no algorithm can achieve a competitive ratio less than 2. This allows us to conclude that the HYBRID algorithm is an optimal online algorithm for cellular networks.

2.1 Upper bound for χ-colorable networks

The HYBRID algorithm for χ-colorable networks can be described as follows.

Preprocessing: Given a χ-colorable network, where the nodes have been colored with χ colors, we partition the frequencies $\{1, 2, 3, \ldots\}$ into $\chi + 1$ disjoint subsets, F_i for $i = 0, 1, \ldots, \chi$.

$$\begin{aligned} F_0 &= \{1, \chi+2, 2\chi+3, \ldots\} \\ F_1 &= \{2, \chi+3, 2\chi+4, \ldots\} \\ F_2 &= \{3, \chi+4, 2\chi+5, \ldots\} \\ &\vdots \\ F_\chi &= \{\chi+1, 2\chi+2, 3\chi+3, \ldots\} \end{aligned}$$

Frequency Assignment Scheme: For each new call, supposing that it emanates from a node v with color x ($1 \leq x \leq \chi$), we assign a frequency to the call either from F_0 or F_x according to the following scheme.

1. Let y be the smallest number in F_0 such that frequency y is not assigned to any call from v or neighbors of v.
2. Let z be the smallest number in F_x such that frequency z is not assigned to any call from v.
3. Assign frequency $\min\{y, z\}$ to the new call.

The HYBRID algorithm is in fact a hybrid combination of ideas behind FAA and GREEDY. Like FAA, HYBRID first partitions the frequencies into disjoint sets; however, HYBRID divides the frequencies into $\chi + 1$ sets instead of χ, with a special set F_0 whose frequencies may be used by calls from any node. While calls from a given node are assigned frequencies from a single set in FAA, HYBRID assigns frequencies from two sets F_0 and F_x to calls from a given node with color x. The greedy approach is taken when selecting the particular frequency to assign from the two sets.

For example, consider a cellular network as shown in Figure 2, in which the cells are labeled from 1 to 21, and also colored by "RGB" scheme, such that no two adjacent cells are colored with the same color. A *call sequence* $\sigma = (1, 5, 7, 1)$ means calls are emanated from cells

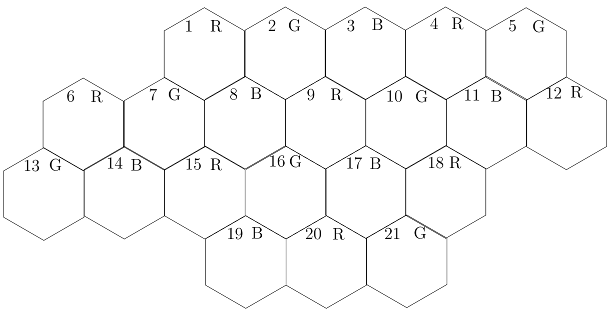

Figure 2: network structure with RGB color scheme.

$1, 5, 7, 1$ respectively in time order. As the time differences between any two subsequent calls can be arbitrarily small or large, simultaneous calls can be represented by sequential calls with arbitrarily small time separation. We use a *frequency sequence* $[1, 1, 3, 2]$ to denote the frequencies assigned by the online algorithm to the calls of σ, that is, the online algorithm assigns the frequencies $1, 1, 3, 2$ to calls from the cells $1, 5, 7, 1$, respectively.

Let us consider a call sequence $\sigma_1 = (8, 1, 6, 10, 8, 4, 8, 19, 6, 8, 13, 11)$. To apply the algorithm Hybrid , we first partition the frequencies into the following sets:

$$\begin{aligned} F_0 &= \{1,\ 5,\ 9, \ldots, 4i+1, \ldots\} \\ F_1 &= \{2,\ 6, 10, \ldots, 4i+2, \ldots\} \\ F_2 &= \{3,\ 7, 11, \ldots, 4i+3, \ldots\} \\ F_3 &= \{4,\ 8, 12, \ldots, 4i+4, \ldots\} \end{aligned}$$

F_0 is the shared frequency set, which can be used in any cell. F_1 can only be used in cells with color R, F_2 in cells with color G and F_3 in cells with color B.

Now apply the algorithm Hybrid to the above mentioned instances σ_1. The frequencies assigned by Hybrid are $[1, 2, 1, 1, 4, 2, 5, 1, 2, 8, 3, 4]$, as shown in Figure 3. The span of frequencies used by Hybrid is 8 and it is easy to verify that the optimal algorithm uses at least 5. In this case the performance ratio is 1.6.

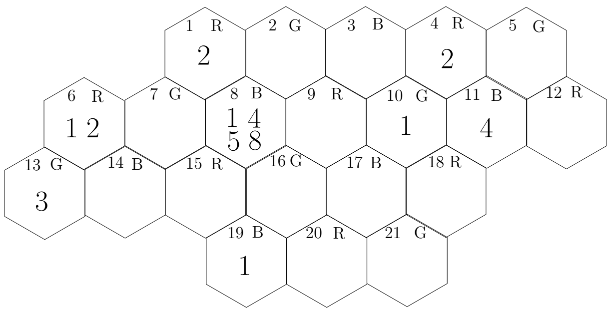

Figure 3: Hybrid works the call sequence $(8, 1, 6, 10, 8, 4, 8, 19, 6, 8, 13, 11)$

Theorem 1. *The competitive ratio of the* Hybrid *algorithm for χ-colorable networks is $(\chi + 1)/2$.*

Proof. Let h be the highest frequency used by Hybrid on a χ-colorable network. Suppose frequency h is assigned to a call C from a node v with color x. When call C arrives, Hybrid considers the two frequencies as follows.

1. $y = (\chi + 1)i + 1 \in F_0$ for some integer $i \geq 0$, implying that there were i calls emanating from node v or neighbors of node v that had already been assigned a frequency from F_0. Either all i of these calls emanate from node v, or there exists a maximum integer q such that $0 \leq q < i$ and frequency $(\chi + 1)q + 1 \in F_0$ is assigned to a particular neighbor of node v, say v' colored x'. In the latter case, we can conclude that there are at least i calls emanating from node v and the neighbor v' since:

   (a) the number of calls from v' assigned a frequency from $F_{x'}$ is at least q (namely, those calls assigned frequencies spanning $(x' + 1)$ to $(\chi + 1)(q - 1) + (x' + 1)$); and

   (b) the number of calls from v' assigned a frequency from F_0 is at least 1 (namely, the call assigned frequency $(\chi + 1)q + 1$); and

   (c) the number of calls from v assigned a frequency from F_0 is at least $i - q - 1$ (namely, those calls assigned frequencies spanning $(\chi + 1)(q + 1) + 1$ to $(\chi + 1)(i - 1) + 1$).

2. $z = (\chi + 1)j + (x + 1) \in F_x$ for some integer $j \geq 0$, implying that there were j calls emanating from node v that had already been assigned a frequency from F_x.

Including call C, we can conclude that there are at least $i + j + 1$ calls (from F_0, F_x and $F_{x'}$) emanating from node v and the neighbor v' (colored x') of v.

There are two cases to consider:

- Case 1: if $h \in F_0$ then $y < z$ or $(\chi + 1)i + 1 < (\chi + 1)j + (x + 1)$ or $j \geq i$
- Case 2: if $h \in F_x$ then $z < y$ or $(\chi + 1)j + (x + 1) < (\chi + 1)i + 1$ or $i \geq j + 1$

For Case 1, the argument proceeds as follows:

1. Since $j \geq i$, there are at least $2i + 1$ calls emanating from node v or its neighbor v'.
2. If there are at least $2i+1$ calls emanating from node v or its neighbor v', any optimal off-line frequency assignment must use a span of frequencies at least $2i + 1$ to avoid interference.
3. The competitive ratio is therefore at most $h/(2i+1) = ((\chi+1)i+1)/(2i+1) \leq (\chi+1)/2$ because $(\chi + 1) \geq 2$.

For Case 2, the argument proceeds as follows:

1. Since $i \geq j + 1$, there are at least $2j + 2$ calls emanating from node v or its neighbor v'.
2. If there are at least $2j + 2$ calls emanating from node v or its neighbor v', any optimal off-line frequency assignment must use a span of frequencies at least $2j + 2$ to avoid interference.
3. The competitive ratio is therefore at most $h/(2j + 2) = ((\chi + 1)j + (x + 1))/(2j + 2) \leq (\chi + 1)/2$ because $x \leq \chi$.

In both cases, the competitive ratio is at most $(\chi + 1)/2$. □

Since the cellular networks are 3-colorable, we have the following corollary.

Corollary 2. *The competitive ratio of* Hybrid *for cellular networks is* 2.

2.2 Lower bound for cellular networks

We show that Hybrid is optimal for cellular networks by giving a matching lower bound. Precisely, we construct (using an adversary) a problem instance in which no online algorithm can achieve a competitive ratio less than 2.

Theorem 3. *No online algorithm for cellular networks has a competitive ratio less than* 2.

Proof. Given any online algorithm $\mathcal{A}$, consider a cellular network with cells labeled $a, b, c, d, e, f, g, h, i, j$ and k as shown in Figure 4.

The adversary runs as follows. In the first step, calls are made from cells a, b, j and k. Algorithm $\mathcal{A}$ must assign the same single frequency, say 1, to all of these calls; otherwise, the adversary stops and the competitive ratio of $\mathcal{A}$ will be at least 2.

In the second step, a new call is made from each of cells c and e. If algorithm $\mathcal{A}$ assigns the same frequency to both calls, without loss of generality, say 2, then the adversary will cause a new call from each of cells g and h. These two new calls require two new frequencies, and thus a span of 4 frequencies is used by $\mathcal{A}$. Given that the optimal off-line algorithm needs only a span of 2 frequencies, $\mathcal{A}$'s competitive ratio is then 2.

However, if algorithm $\mathcal{A}$ assigns different frequencies to calls from cells c and e, say 2 and 3, the adversary will proceed to make a new call from each of cells f and i. Algorithm $\mathcal{A}$ must then assign 3 to the call from cell f and 2 to the call from cell i; otherwise, the adversary will stop with algorithm $\mathcal{A}$ having used a span of 4 frequencies when only 2 were needed. The adversary will continue with a new call from each of cells d, g and h, whereupon algorithm $\mathcal{A}$ must assign three new frequencies to the three new calls. By now, algorithm $\mathcal{A}$ has used at least a span of 6 frequencies. Given that the optimal off-line algorithm can use only 3, $\mathcal{A}$'s competitive ratio is again 2.

Therefore, at best, $\mathcal{A}$'s competitive ratio is 2. □

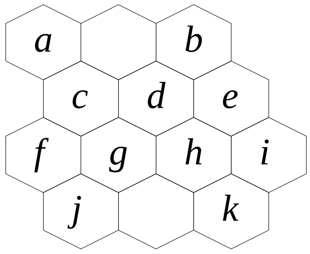

Figure 4: The cellular network for proving the lower bound.

3 Asymptotic bounds for cellular networks

In Section 2 we give an algorithm that achieves a competitive ratio of 2 for cellular networks and show that no algorithm can achieve a competitive ratio less than 2. So it would seem that the problem for cellular networks is completely solved and no further study is necessary.

However, when more and more calls per cell are made: the (asymptotic) bounds for the competitive ratio can be proved to fall below the absolute bound of 2. This is a phenomenon that does exist for some problems. For example, in the priority list online scheduling for n bounded size independent jobs on 2 and 3 machines, the competitive ratio is tight at 3/2 and 4/3, respectively, when n is small but approaches 1 when n is large.

To understand this phenomenon for the frequency allocation problem, we need to re-examine the lower bound proofs. For example, in the lower bound proof for cellular networks, the adversary creates a worst case scenario in which the optimal off-line algorithm requires only 2 or 3 frequencies, while any online algorithm is forced to use at least 4 or 6 frequencies, giving therefore a competitive ratio of 2. The critical step occurs when the online algorithm is presented with one call emanating from each of two non-neighboring cells (separated by two cells) arriving at the same time. The algorithm makes a decision whether these two calls will be assigned the same new frequency or different new frequencies. Choosing different frequencies would mean that at least 2 frequencies are used and it becomes clear that the online algorithm would make the wrong choice if the adversary stopped further calls because the optimal solution would only use one frequency. The competitive ratio is 2. On the other hand, choosing the same frequency would compel the adversary to continue to make more calls in order to prove the online algorithm's choice was flawed and a competitive ratio of 2. Note that this choice of whether to assign the same or different frequencies to two calls is a discrete one. However, when dealing with a large number of calls per cell (instead of one call per non-neighboring cell), the choice can be less discrete. For example, we can choose to assign the same frequency to half of the calls and a different frequency to the other half of the calls to the dismay of the adversary. It is this concept that allows a breakthrough on the lower bound.

As it turns out, we can modify the Hybrid algorithm to achieve better performance when dealing with a large number of calls in cellular networks. Recall that for cellular networks Hybrid divides the spectrum of frequencies into 4 disjoint sets, one of which may be used in any cells. This set is called the *shared* set. As HYBRID is a combination of the fixed allocation algorithm (FAA) and the greedy algorithm (GREEDY), we observe that when all calls emanate from a single isolated cell, GREEDY gives the optimal solution. So ideally, we would want the greedy component to play a bigger role when the number of calls is large. This is possible if we increase the size of the shared set with respect to the other sets. The question is: what is the right size for the shared set? Hence, our modified algorithm has effectively two parameters, the standard size of the sets and the size of the larger shared set. The appropriate sizes to achieve the best upper bound are derived.

3.1 Asymptotic upper bound

We propose a family of HYBRID algorithms characterized by two integer parameters α and β, where both α and β are non-negative and at least one of them is non-zero. The two parameters enable us to control the "degree" which we combine FAA and GREEDY. In the extreme case, when $\alpha = 0$, HYBRID becomes a pure FAA. On the other hand, when $\beta = 0$, HYBRID becomes a pure GREEDY. The particular HYBRID algorithm in Section 2 has $\alpha = 1$ and $\beta = 1$. In the following we give a general description of the family of HYBRID algorithms.

Let $\Delta = \alpha + 3\beta$. Conceptually, frequencies are divided into groups of Δ frequencies. A frequency f is in group i if $i\Delta < f \leq (i+1)\Delta$ for $i \geq 0$. The online algorithm partitions the set of all frequencies $\{1, 2, 3, \ldots\}$ into four disjoint subsets, F_0, F_1, F_2 and F_3. Subset F_0 receives α frequencies from each group while each of F_1, F_2 and F_3 receives β. If we are focusing on the asymptotic behavior of the online algorithms, as long as the proportion of frequencies from each group among the subsets are fixed, the exact distribution of the Δ frequencies from each group to the four subsets does not affect the asymptotic performance of the algorithm. For instance, consider the following particular distribution of frequencies from group i to the four subsets. Let $\gamma = \min\{\alpha, \beta\}$.

$$
\begin{aligned}
F_0 &= \{i\Delta + 1, i\Delta + 5, \ldots, i\Delta + 4\gamma - 3\} \cup \\
&\quad \{i\Delta + 4\gamma + j \mid 1 \leq j \leq \alpha - \beta\} \\
F_1 &= \{i\Delta + 2, i\Delta + 6, \ldots, i\Delta + 4\gamma - 2\} \cup \\
&\quad \{i\Delta + 4\gamma - 2 + 3j \mid 1 \leq j \leq \beta - \alpha\} \\
F_2 &= \{i\Delta + 3, i\Delta + 7, \ldots, i\Delta + 4\gamma - 1\} \cup \\
&\quad \{i\Delta + 4\gamma - 1 + 3j \mid 1 \leq j \leq \beta - \alpha\} \\
F_3 &= \{i\Delta + 4, i\Delta + 8, \ldots, i\Delta + 4\gamma\} \cup \\
&\quad \{i\Delta + 4\gamma + 3j \mid 1 \leq j \leq \beta - \alpha\}
\end{aligned}
$$

The family of HYBRID algorithms assigns a frequency to a new call using the same frequency assignment scheme as used in Section 2.1.

There is a simple property of HYBRID which is useful for analysis.

Lemma 4. *If a frequency of group k is assigned by a cell colored c, then the number of frequencies of F_c assigned by the cell is at least βk.*

Proof. All the frequencies of F_c in group i for $0 \leq i \leq k-1$, which are lower than any frequency in group k, must have been assigned to calls from the cell. Since there are βk such frequencies, the lemma follows. □

We show that the asymptotic competitive ratio of HYBRID approaches 1.9126 when β/α approaches 0.8393. First, we have the following lemma to lower bound the span of frequencies required by the optimal off-line algorithm, which is the total number of calls emanating from three mutually adjacent cells.

Lemma 5. *If a cell A assigns a frequency from group k, then the total number of calls from cell A and two of its neighbors, which are also adjacent to each other, is at least $(\alpha+\beta)k - \beta(1+\frac{1}{\alpha})$ for $\frac{\beta}{\alpha} \geq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$ (≈ 0.8393), and at least $(\frac{\beta^3}{\alpha^2} + \frac{2\beta^2}{\alpha} + \beta - \alpha)k - \beta(1+\frac{1}{\alpha})$ for $\frac{\beta}{\alpha} \leq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$ (≈ 0.8393).*

Proof. Let $B_1, B_2, \ldots, B_6$ denote the six neighbors of cell A in clockwise order as shown in Figure 5. We divide the six neighbors into three groups $\{B_1, B_2\}$, $\{B_3, B_4\}$, and $\{B_5, B_6\}$.

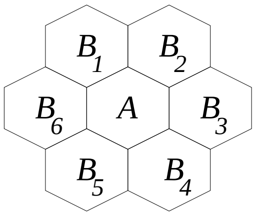

Figure 5: Cell A and its neighboring cells.

Let F_0' be the subset of F_0 containing all the frequencies from group 0 to group $k-1$, precisely, $F_0' = \{f \mid f \in F_0 \text{ and } f \leq k\Delta\}$. Thus, $|F_0'| = \alpha k$. Since a frequency from group k is assigned by cell A, by the algorithm, each frequency in F_0' should have already been assigned by cell A or its neighbors, cells $B_1, B_2, \ldots, B_6$. Assume that x of them are assigned by cell A, y by cells B_1 and B_2, z by cells B_3 and B_4, and at least $\alpha k - x - y - z$ by cells B_5 or B_6. Without loss of generality, we can assume that the highest frequency of F_0' assigned by cell B_1 or B_2 is at least that of F_0' assigned by cell B_3 or B_4, and the highest frequency of F_0' assigned by cell B_3 or B_4 is at least that of F_0' assigned by cell B_5 or B_6.

Suppose the color of cell A is c_1, cells B_1, B_3 and B_5 are c_2, and cells B_2, B_4 and B_6 are c_3. Consider the total number of calls from cells A, B_1 and B_2.

- There are βk frequencies of F_{c_1} from groups 0 to $k-1$ assigned by cell A.
- By the assumption, there are $x + y$ frequencies of F_0' assigned by cells A, B_1 and B_2.
- We claim that there are at least $\beta\lfloor(\alpha k - x - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_1 or B_2. The claim is true because $\alpha k - x$ frequencies in F_0' are assigned by cells B_1, B_2, B_3, B_4, B_5 and B_6, the highest frequency among these frequencies, which is assigned by B_1 or B_2, must be from group $\lfloor(\alpha k - x - 1)/\alpha\rfloor$ or above. Thus by Lemma 4, the claim follows.

Note that all these frequencies are from disjoint sets. The total number of calls from cells A, B_1 and B_2 is at least

$$\begin{aligned} T_1 &= \beta k + x + y + \beta\lfloor(\alpha k - x - 1)/\alpha\rfloor \\ &\geq 2\beta k + (1 - \beta/\alpha)x + y - \beta(1 + 1/\alpha). \end{aligned}$$

Consider the total number of calls from cells A, B_3 and B_4.

- There are βk frequencies of F_{c_1} from groups 0 to $k-1$ assigned by cell A.
- By the assumption, there are $x + z$ frequencies of F_0' assigned by cells A, B_3 and B_4.
- We claim that there are at least $\beta\lfloor(\alpha k - x - y - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_3 or B_4. The claim is true because $\alpha k - x - y$ frequencies of F_0' are assigned by cells B_3, B_4, B_5 and B_6, the highest frequency among these frequencies, which is assigned by B_3 or B_4, must be from group $\lfloor(\alpha k - x - y - 1)/\alpha\rfloor$ or above. Thus by Lemma 4, the claim follows.

Note that all these frequencies are from disjoint sets. The total number of calls from cells A, B_3 and B_4 is at least

$$\begin{aligned} T_2 &= \beta k + x + z + \beta\lfloor(\alpha k - x - y - 1)/\alpha\rfloor \\ &\geq 2\beta k + (1-\beta/\alpha)x - (\beta/\alpha)y + z - \beta(1+1/\alpha). \end{aligned}$$

Consider the total number of calls from cells A, B_5 and B_6.

- There are at least βk frequencies of F_{c_1} from groups 0 by $k-1$ assigned by cell A.
- By the assumption, there are at least $\alpha k - y - z$ frequencies of F'_0 assigned by cells A, B_5 and B_6.
- We claim that there are at least $\beta\lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor$ frequencies of F_{c_2} or F_{c_3} assigned by cells B_5 or B_6. The claim is true because $\alpha k - x - y - z$ frequencies from F'_0 are assigned by cell B_5 and B_6, the highest of which must be from group $\lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor$ or above. Hence, by Lemma 4, the claim follows.

Note that all these frequencies are from disjoint sets. The total number of calls from cells A, B_5 and B_6 is at least

$$\begin{aligned} T_3 &= \beta k + \alpha k - y - z + \beta\lfloor(\alpha k - x - y - z - 1)/\alpha\rfloor \\ &\geq (\alpha+2\beta)k - (\beta/\alpha)x - (1+\beta/\alpha)y - (1+\beta/\alpha)z - \beta(1+1/\alpha). \end{aligned}$$

The total number of calls from cells A and two of its neighbors, which are also adjacent to each other, is bounded below by the value $\max\{T_1, T_2, T_3\}$. It can be verified that when $z = (1+\beta/\alpha)y$ and $x = \alpha k - y(3 + 3\beta/\alpha + \beta^2/\alpha^2)$, $\max\{T_1, T_2, T_3\}$ achieves the minimum value, where the values of T_1, T_2 and T_3 are all equal, i.e.,

$$\max\{T_1, T_2, T_3\} = (\alpha+\beta)k - \beta(1+\frac{1}{\alpha}) + y(\frac{\beta^3}{\alpha^3} + \frac{2\beta^2}{\alpha^2} - 2).$$

For $\frac{\beta}{\alpha} \geq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$, we have $\frac{\beta^3}{\alpha^3} + \frac{2\beta^2}{\alpha^2} - 2 \geq 0$. Therefore, $\max\{T_1, T_2, T_3\} \geq (\alpha+\beta)k - \beta(1+\frac{1}{\alpha})$. On the other hand, since $y \leq \alpha k$ (the number of frequencies of F'_0 assigned by cells B_1 and B_2 is at most αk), when $\frac{\beta}{\alpha} \leq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$, we have $\frac{\beta^3}{\alpha^3} + \frac{2\beta^2}{\alpha^2} - 2 \leq 0$, and hence $\max\{T_1, T_2, T_3\} \geq (\alpha+\beta)k - \beta(1+\frac{1}{\alpha}) + \alpha k(\frac{\beta^3}{\alpha^3} + \frac{2\beta^2}{\alpha^2} - 2) = (\frac{\beta^3}{\alpha^2} + \frac{2\beta^2}{\alpha} + \beta - \alpha)k - \beta(1+\frac{1}{\alpha})$. As a result, the lemma follows. □

By the above lemma, if we set the values of α and β such that $\frac{\beta}{\alpha} \geq -\frac{2}{3} + \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3}$, Hybrid has the asymptotic competitive ratio of $(\alpha+3\beta)/(\alpha+\beta)$.

Theorem 6. *The asymptotic competitive ratio of* Hybrid *for cellular networks is* $(\alpha+3\beta)/(\alpha+\beta)$ *for* $\frac{\beta}{\alpha} \geq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$ (≈ 0.8393).

Proof. Suppose the highest frequency used by Hybrid is from group k assigned by a cell A, which is at most $(k+1)\Delta = (\alpha+3\beta)(k+1)$. By Lemma 5, for $\frac{\beta}{\alpha} \geq \frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$, the total number of calls from cell A and two of its neighbors, which are also adjacent to each other, is at least $(\alpha+\beta)k - \beta(1+1/\alpha)$, and hence, the span of frequencies used must be at least $(\alpha+\beta)k - \beta(1+1/\alpha)$. Therefore, the asymptotic competitive ratio of Hybrid is

$$R^\infty \leq \lim_{k\to\infty} \frac{(\alpha+3\beta)(k+1)}{(\alpha+\beta)k - \beta(1+1/\alpha)} = \frac{\alpha+3\beta}{\alpha+\beta}.$$

□

Corollary 7. *The asymptotic competitive ratio of* Hybrid *for cellular networks approaches* $\frac{11}{3} + \frac{2(19-3\sqrt{33})^{1/3}}{9} - \frac{2(19-3\sqrt{33})^{2/3}}{9} + \frac{2(19+3\sqrt{33})^{1/3}}{9} - \frac{2(19+3\sqrt{33})^{2/3}}{9}$ (≈ 1.9126).

Proof. As we can assign integer values to α and β such that β/α is arbitrarily close to $\frac{\sqrt[3]{19+3\sqrt{33}}}{3} + \frac{\sqrt[3]{19-3\sqrt{33}}}{3} - \frac{2}{3}$, we have $(\alpha+3\beta)/(\alpha+\beta)$ arbitrarily close to $\frac{11}{3} + \frac{2(19-3\sqrt{33})^{1/3}}{9} - \frac{2(19-3\sqrt{33})^{2/3}}{9} + \frac{2(19+3\sqrt{33})^{1/3}}{9} - \frac{2(19+3\sqrt{33})^{2/3}}{9}$. □

3.2 Guaranteed optimal absolute competitive ratio

In practice, it is preferable to have small values of α and β while the performance can be maintained. For example, we can set the values of α and β to 13 and 11, respectively. By Theorem 6, this particular Hybrid algorithm has an asymptotic competitive ratio $23/12 \approx 1.9167$. In the following theorem, we also prove that at the same time this algorithm achieves the optimal absolute competitive ratio, i.e., 2, by a similar but more conservative analysis as in Lemma 5.

Theorem 8. *For $\alpha = 13$ and $\beta = 11$, the (absolute) competitive ratio of* Hybrid *for cellular networks is* 2.

Proof. As we are deriving the absolute competitive ratio of the Hybrid algorithm, we have to fix the distribution of the Δ frequencies of a group to the four subsets. In particular, we follow the distribution stated in Section 3, i.e., for frequencies of group k,

$$\begin{aligned} F_0 &= \{46k+1, 46k+5, \ldots, 46k+41\} \\ &\quad \cup\{46k+45, 46k+46\} \\ F_1 &= \{46k+2, 46k+6, \ldots, 46k+42\} \\ F_2 &= \{46k+3, 46k+7, \ldots, 46k+43\} \\ F_3 &= \{46k+4, 46k+8, \ldots, 46k+44\} \end{aligned}$$

Assume that the highest frequency assigned by Hybrid is for a cell A and the frequency is from group $k \geq 1$. If $k = 0$, the scenario is similar to the case for $\alpha = \beta = 1$ where we proved a competitive ratio of 2 in Section 2.1. Let the highest frequency be $h = 46k + 4i - 3 + c + j$ for some integers $1 \leq i \leq 11$ and $j \in \{0, 4, 5\}$ and $0 \leq c \leq 3$. Note that c represents the color to which the frequency belongs (color 0 to represent F_0), and $j = 4$ or 5 if the frequency is either $46k + 45$ or $46k + 46$ that belongs to F_0. The core part of the proof consists of a more conservative analysis than that in Lemma 5 to lower bound the number of calls from cell A and two of its neighbors, and hence show that the competitive ratio of Hybrid is 2.

Similar to the proof of Lemma 5, let $B_1, B_2, \ldots, B_6$ denote the six neighbors of cell A in clockwise order as shown in Figure 5. The color of cell A is c_1, cells B_1, B_3 and B_5 are c_2, and cells B_2, B_4 and B_6 are c_3. The six neighbors are divided into three groups $\{B_1, B_2\}$, $\{B_3, B_4\}$, and $\{B_5, B_6\}$.

Let F'_0 be a subset of F_0 containing the frequencies at most h, precisely, $F'_0 = \{f \mid f \in F_0 \text{ and } f \leq h\}$. Thus, $|F'_0| = \alpha k + i + j' = 13k + i + j'$ where $j' = \max\{0, j - 3\}$. By the algorithm, each frequency in F'_0 should have already been assigned by cell A or its neighbors, cells $B_1, B_2, \ldots, B_6$. Assume that x of them are assigned by cell A, y by cells B_1 and B_2, z by cells B_3 and B_4, and at least $13k + i + j' - x - y - z$ to cells B_5 or B_6. Without loss of generality, we can assume that the highest frequency of F'_0 assigned by cell B_1 or B_2 is at least that assigned by cell B_3 or B_4, and the highest frequency of F'_0 assigned by cell B_3 or B_4 is at least that assigned by cell B_5 or B_6.

Consider the total number of calls from cells A, B_1 and B_2.

- There are at least $11k+i-c'$ frequencies of F_{c_1} assigned by cell A, where $c'=1$ if $c=0$, i.e., the highest frequency assigned is from F_0, and $c'=0$ if otherwise.
- By the assumption, there are $x+y$ frequencies of F'_0 assigned by cells A, B_1 and B_2.
- We claim that there are at least $11(13k+i+j'-x-1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_1 or B_2. As $13k+i+j'-x$ frequencies of F'_0 are assigned by $B_1, B_2, \ldots, B_6$, and by the assumption the highest frequency among these frequencies is assigned by cell B_1 or B_2, following the HYBRID algorithm, there must have been at least $11(13k+i+j'-x-1)/13$ frequencies either of F_{c_2} or F_{c_3} assigned by B_1 or B_2, respectively.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_1 and B_2 is at least

$$\begin{aligned} T_1 &= 11k+i-c'+x+y+11(13k+i+j'-x-1)/13 \\ &= 22k+24i/13+11j'/13+2x/13+y-c'-11/13. \end{aligned}$$

Consider the total number of distinct frequencies (calls) from cells A, B_3 and B_4.

- There are at least $11k+i-c'$ frequencies of F_{c_1} assigned by cell A.
- By assumption, there are $x+z$ frequencies of F'_0 assigned by cells A, B_3 and B_4.
- Similarly to the previous case, it can be verified that there are at least $11(13k+i+j'-x-y-1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cell B_3 or B_4.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_3 and B_4 is at least

$$\begin{aligned} T_2 &= 11k+i-c'+x+z+11(13k+i+j'-x-y-1)/13 \\ &= 22k+24i/13+11j'/13+2x/13-11y/13+z-c'-11/13. \end{aligned}$$

Consider the total number of calls from cells A, B_5 and B_6.

- There are at least $11k+i-c'$ frequencies of F_{c_1} assigned by cell A.
- By assumption, there are at least $x+13k+i+j'-x-y-z$ frequencies of F'_0 assigned by cells A, B_5 and B_6.
- Similar to the previous case, it can be verified that there are at least $11(13k+i+j'-x-y-z-1)/13$ frequencies of F_{c_2} or F_{c_3} assigned by cells B_5 or B_6.

Note that all these frequencies are from disjoint sets. Thus the total number of calls from cells A, B_5 and B_6 is at least

$$\begin{aligned} T_3 &= 11k+i-c'+13k+i+j'-y-z+11(13k+i+j'-x-y-z-1)/13 \\ &= 35k+37i/13+24j'/13-11x/13-24y/13-24z/13-c'-11/13. \end{aligned}$$

The total number of calls from cells A and two of its neighbors, which are also adjacent to each other, is bounded below by the value $\max\{T_1, T_2, T_3\}$. It can be verified that when $z = 24y/13$ and $x = 13k+i+j'-1057y/169$, $\max\{T_1, T_2, T_3\}$ achieves the minimum value, i.e.,

$$\begin{aligned} \max\{T_1, T_2, T_3\} &= 24k+2i+j'+83y/2197-c'-11/13 \\ &\geq 24k+2i+j'-c'-11/13. \end{aligned}$$

As a result the competitive ratio of HYBRID is at most

$$\begin{aligned} & \frac{46k+4i-3+c+j}{24k+2i+j'-c'-11/13} \\ \leq\ & \frac{46(k-1)+46+4i+j-(3-c)}{24(k-1)+23+2i+\max\{0,j-3\}-\max\{0,1-c\}} \\ \leq\ & 2 \end{aligned}$$

(because $k \geq 1$, $j \in \{0,4,5\}$, $c \in \{0,1,2,3\}$ and $c' = \max\{0, 1-c\}$). □

3.3 Asymptotic lower bound

We give a lower bound of 1.5 on the asymptotic competitive ratio for cellular networks.

Theorem 9. *No online algorithm for cellular networks has an asymptotic competitive ratio less than* $3/2$.

Proof. Given any online algorithm $\mathcal{A}$, consider the cellular network shown in Figure 6. The adversary consists of three steps.

In step 1, the adversary makes n calls in each of the cells with label A_j $(j = 1, 2, 3, 4)$. Let xn be the minimal number of common frequencies in any two cells labeled with A_j. Then the online algorithm uses at least $(2-x)n$ distinct frequencies while the optimal off-line algorithm uses at least n distinct frequencies. Therefore, the asymptotic competitive ratio in this step is at least $2-x$. The adversary stops the sequence of calls if $x \leq 1/2$.

Otherwise, in step 2, the adversary makes n calls in each of the cells with label B_j $(j = 1, \ldots, 6)$. Consider the four cells A_1, B_1, B_2, A_3. The number of distinct frequencies used by the algorithm $\mathcal{A}$ is at least $(2+x)n$. We assume that in this step, there are a total of $(2+x+y)n$ frequencies used by the algorithm $\mathcal{A}$. The adversary stops the sequence of calls if $x+y \geq 1$, which implies an asymptotic competitive ratio of at least $3/2$.

Otherwise, in step 3, the adversary makes n calls in each of the cells with label C_j $(j = 1, 2, 3)$. It is worth noting that the total distinct frequencies used by cells B_1, B_2, A_3 is at least $(2+x)n$, since B_1 reuses at most $(1-x)n$ frequencies from A_3 and the frequencies used by B_1 should be totally disjoint from the frequencies used by cell B_2. Thus, cell C_1 reuses at most $(2+x+y)n - (2+x)n = yn$ old frequencies. This ensures that C_1 has to use at least $(1-y)n$ new frequencies. By a similar argument, one can check that cells C_2 and C_3 will contribute another $2(1-y)n$ new distinct frequencies. Thus, there are totally $(2+x+y)n + 3(1-y)n$ distinct frequencies used by algorithm $\mathcal{A}$. However, the optimal algorithm uses at most $3n$. Hence, $R_{\mathcal{A}}^{\infty} \geq (5+x-2y)/3$. Since $y < 1/2$ and $x > 1/2$, we get $R_{\mathcal{A}}^{\infty} \geq 3/2$. □

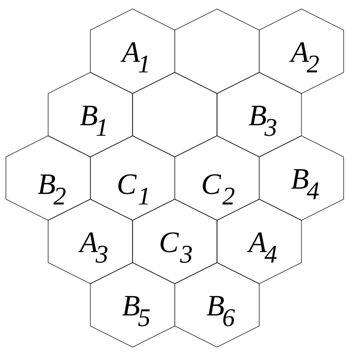

Figure 6: The cellular network for proving the asymptotic lower bound.

4 Concluding Remarks

Many interesting problems on the online frequency allocation remain open. One of them is a generalization of the problem studied in this paper that considers calls may terminate at any time [4,13] and the occupied frequency is released when a call terminates. Both fixed allocation and greedy algorithms yield a competitive ratio 3 in this case [5] . No algorithm is known to be better than 3-competitive. Another direction is to generalize the problem by considering more general classes of graphs such as unit disk graphs or planar graphs, instead of hexagon graphs. The frequency allocation problem thus generalized can be seen as a variant of the graph multicoloring problem [18].

Since bandwidth is always a valuable resource, further approach to minimize the span of frequencies used is to allow reassignment of frequencies to some of the existing calls. Thus the aim is to design algorithms with reassignments to achieve lower bandwidth of frequencies when only local information of the neighboring cells is available [7,13,21].

In the third generation (3G) mobile communication, Orthogonal Variable Spreading Factor (OVSF) code assignment is a fundamental problem in Wideband Code-Division Multiple-Access (W-CDMA) systems. In the OVSF problem, codes must be assigned to incoming call requests with different data rate requirements from a complete binary code tree, which is more general than frequency allocation. Erlebach et al. [9] gave the first h-competitive algorithm which minimizes the number of code assignments/reassignments, where h is the height of the code tree. With the help of extra bandwidth, Chin et al. [8] gave a 5-competitive algorithm, Furthermore, they proposed a constant competitive algorithm [6] with no resource augmentation. There is still a big gap between the lower and upper bound of the competitive ratio. How to close the gap becomes an interesting problem for further research.

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