A New Approach to Quantum Logic

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Chapter 1

Introduction

The main purpose of this monograph is to present the new approach to quantum logic developed by the authors in recent years in the coherent form of a book. This approach constitutes a new way of looking at the connection between quantum mechanics and logic.

Only part of the ideas and the material presented here has been published so far. The published material consists of "Quantum logic, Hilbert space, revision theory" by Engesser and Gabbay in *Artificial Intelligence*, 2002 and "Algebras of Measurements: the logical structure of Quantum mechanics" by Lehmann-Engesser-Gabbay in the *International Journal of Theoretical Physics*, 2006. These publications together with several preprints and drafts form the basis of a more extensive theory which was developed collaboratively during the last two years. This general theory is to form the core of the monograph.

The message of the book is of interest to a broad audience consisting of logicians, mathematicians, philosophers of science, researchers in Artificial Intelligence and last but not least physicists. These communities, however, strongly differ in their scientific backgrounds. Normally, a physicist has no training in mathematical logic, and a logician is by no means expected to master the Hilbert space formalism of quantum mechanics. This fact constitutes a major problem in any attempt to present the topic of quantum logic in a way accessible to the broad audience to which, in principle, it is of interest. In a journal article for instance it is extremely difficult, if not impossible, to solve this problem. The prime intention of the authors is, apart from giving an extensive and coherent account of their theory, to present their approach in such a way that it is accessible to the heterogeneous audience described.

Therefore, the first chapters serve to provide the reader with the logical and mathematical background that will enable him to understand the subsequent chapters. Chapter 2 presents the prerequisites from logic. In chapter 3 we introduce the concept of a Hilbert space, which constitutes the core structure of the formalism of quantum mechanics. We summarise- for the most part without proof- basic facts of Hilbert space theory which provide the reader with the mathematical equipment essential for the understanding of the core

chapters. Chapter 3, however, contains a first result of our resarch, namely a characterisation of classical Hilbert lattices which is of interest from the purely mathematica point of view.

In Chapter 4 we give a similar summary of the the main facts about the mathematical formalism of quantum mechanics.

Quantum logic has its origin in the famous 1936 now classic paper by Birkhoff-von Neumann entitled "The logic of quantum mechanics". This paper is still today by far the most widely quoted paper in the field. We devote Chapter 5 to a detailed analysis and, in a sense, to a reconstruction of this classic. The reason for this is twofold. The Birkhoff-von Neumann paper is, today, not easy to read, and it is thus an end in itself to interpret and reconstruct it in modern terminology and highlight its main ideas. Moreover, this chapter serves as a basis for putting in perspective the approach to quantum logic put forward in this book. This approach may, to a considerable extent at least, be viewed as a refinement of the ideas of Birkhoff and von Neumann. The relationship between the two views is on the one hand a 'local-global' relationship. In a sense to be made precise the authors' theory may be viewed as the 'local' version of Birkhoff-von Neumann style quantum logic. On the other hand the refinement consists in our view of propositions. In the approach presented in this book the focus is on viewing propositions as projections in Hilbert space rather than (closed) subspaces as do Birkhoff and von Neumann. This allows for a *dynamic view of propositions.*

The core of the message of the book is contained in chapters 6,7,9 and 10. In these chapters we introduce and investigate new concepts which we think can play a fruitful role in quantum logic. Formally, these concepts are abstractions from structures we find in Hilbert space. In chapter 6 we abstract from the lattice of projections of a Hilbert space introducing and studying structures which we call algebras of measurements, M-algebras for short. In coining the term M-algebra for these structures we are aware of the fact that this is a loose way of making use of the term "algebra". In the strict sense of Universal Algebra these structures do not qualify as algebras. Logically, the novelty of this approach consists in a new way of treating propositions. It is inspired by the analogy between propositions and measurements in physics, in particular quantum measurements. Suppose a physicist performs a measurement of a certain physical quantity A pertaining to a certain physical system. Suppose this system is in a certain state x. The physicist will then in general formulate the result of his measurement as a proposition of the form $A = \mu$, where μ is the value of A measured, and he will then claim this proposition to be a true statement about the system under investigation. In this there is no difference between classical and quantum physics. There exists, however, an essential difference between the classical and the quantum case which seems to us of fundamental importance from the logical point of view. Namely, the meanings of the physicist's assertion that $A = \mu$ is true differ in the two cases, classical and quantum. In classical mechanics the proposition $A = \mu$ is a true statement about the physical system in state x. In the quantum case it is a true statement too. The crucial difference, however, is that in the quantum case the proposition

$A = \mu$ is in general no longer a true statement about the state x but about a certain state y distinct from x, namely about the state of the system 'after measurement'. The reason for this is that quantum measurements generally involve, in contrast to 'classical' measurements, a change of state of the system measured. Logically speaking, the situation we have in classical mechanics is this. Given a state x (state of affairs, state of the world...) and some proposition α. Then α has some truth value in state x. In bivalent logic these truth values are 'true' and 'false'. In multi-valued logic there are more truth values, possibly even infinitely many. The situation in quantum mechanics is different. Given a state x and a proposition α. Then α does not necessarily possess *any* truth value in x. Rather it is only in some other state distinct from x , namely in the state 'after measurement', that it acquires a truth value. In Chapter 6 we take this dynamic aspect of quantum propositions seriously. It is the source of inspiration for developing a general logical framework in which the *static* notion of truth of a proposition prevailing in traditional logic is replaced by the more general *dynamic* notion of a proposition acting on states. This framework turns out to be a natural generalisation of the traditional static view in which the logical structure of classical and quantum mechanics can be described and their relationship be put in evidence. Classical logic and correspondingly classical mechanics appear as the *static limiting cases* of a *dynamic framework*.

In chapter 7 we introduce what we call the *local viewpoint in quantum logic* as opposed to the global point of view that prevailed in the preceding chapters. In the Birkhoff-von Neumann paper propositions are represented as sets of states. The concept of a state itself is not the focus of attention. The same is true for our framework of M-algebras as developed in Chapter 6. In that framework states are primitive notions. In chapter 7 we make the concept of a state itself the focus of investigation. This chapter may be viewed as a logical enquiry into the nature of physical states.

Chapter 8 is an interlude addressing primarily those readers who haven't had much contact with quantum mechanics yet. We give a report on some of the well known 'odd' features of the quantum world. Again our intention is twofold. First, the reader can hardly appreciate logical considerations on quantum mechanics without being familiar with the salient physical features of the quantum world. Second, we regard this chapter as a vehicle for conveying the impression to the reader that quantum mechanics touches on fundamental issues, even beyond the realm of physics. It seems that, in contrast to previous physical theories, quantum mechanics raises not just the question what are the laws that govern physical reality but the issue of the very nature of (physical) reality itself. It is often argued both in the seriously philosophical and the popular scientific literature that the proper understanding of quantum mechanics requires a revision of the view of reality to which we are used from classical physics. It is frequently argued that a main obstacle to the proper understanding of quantum mechanics consists in the 'fragmented' world view which underlies classical mechanics and, by the way, also classical logic. The intuition all pervading the liteature is that quantum mechanics requires a more holistic view of reality than we are used to from classical mechanics. Bohm's classic book "Wholeness and

the Implicate Order" is a most profound and eloquent account of this.

In chapter 9 we introduce and study another new concept of crucial importance, namely that of a *holistic logic* and as a special case that of a *Hilbert space logic*. Again, the concept of a holistic logic is an abstraction from logical structures we find in Hilbert space as is the concept of an M-algebra.

In chapter 10 entitled "Towards Hilbert space" we pursue the question whether the concept of a Hilbert space can be characterised in terms of the logical structures studied in the preceding chapters. We present a representation theorem which may be regarded as a positive answer to the above question. This is part of what in the literature on the foundations of quantum mechanics is sometimes called the representation enterprise, a term denoting the project of deriving the formalism of quantum mechanics from certain first principles. In our case these first principles are of a purely logical nature.

The last chapter has a speculative character. There we even permit ourselves a little bit of metaphysics. We speculate about the possible implications our results and considerations presented thus far may have for the treatment of certain foundational problems of quantum mechanics. In particular, we present an admittedly speculative treatment of the measurement problem in which the paradoxical nature this problem displays in other approaches is avoided.

As a rough summary we can say that, conceptually, the message of this book rests on two pillars. The first pillar is the *dynamic view of propositions*. We view propositions as acting on states (of the world) and changing them rather than just being true or false in these states. The second pillar is a *logical enquiry into the nature of physical states*.

The main mathematical results presented in this book arise from these two sources.

Chapter 2

A Crash Course in Logic

Abstract:. The reason for including this chapter is to make the book as self-contained as possible. It should in particular be accessible to physicists, who normally have no training in formal logic. We present the basics of classical propositional logic and non-monotonic logic. In fact, it is possible to provide the reader with all the logical equipment he needs in order to understand the logical investigations in later chapters. The material covered by the introduction to classical propositional logic comprises the following:

The language of classical propositional logic
The notion of a consequence relation, monotonic or not
Presenting classical consequence in (old fashioned) Hilbert style
Truth functional semantics
Soundness and Completeness Theorem
Compactness theorem
Deduction theorem
The Lindenbaum (-Tarski) algebra

This is, in particular, sufficient to enable the reader to understand the logical investigations in the chapters 6, 7 $94 and 10$.

Basics of Nonmonotonic Logic ■

2.1 Basics of Classical Propositional Logic

All the reader needs to know about classical propositional logic is:

- *The Deduction Theorem*
- *The Soundness and Completeness Theorem*
- *The Compactness Theorem*
- *The Concept of a Lindenbaum (-Tarski) Algebra*

2.1.1 Introductory Remarks

Logic nowadays means formal logic. Modern logic studies logical systems as *formal systems* based on a precisely defined *formal language*. The concept most central to logic is that of *logical consequence*. Logical consequence is a relation between two statements α and β or, more generally, a relation between a set of statements Σ and a statement α. One may synonymously say "α is a logical consequence of Σ" or "α follows (logically) from Σ" or "α can be deduced logically (or is deducible) from Σ". Logical deduction is a vital part of our competence as human beings in both everyday and scientific discourse, and it is one of the seminal achievements of modern mathematical logic to have provided the tools for a mathematically rigorous analysis of the intuitive concept of logical consequence.

Given two statements α and β in some (natural or formal) language. What does it mean to say that β is a logical consequence of α? The first idea that may come to mind is to say that β can, in some way, be *proved* from α in the sense that if α is assumed then we can deduce β using certain rules of logical deduction. On this view "β follows from α" means "β is provable from α". It is thus obvious that a rigorous analysis must then provide a precise definition of what it means to say "is provable". In other words, the logician's task then consists in making precise the concept of a *proof.*

Another natural intuition in approaching the issue of logical consequence is this. We may say that "β follows from α" means something like "Whenever α is true, then so is β". In this case a rigorous treatment requires a 'theory of truth'.

In fact, modern (formal) logic reflects these natural intuitions in the way it explicates and studies logical consequence.

Generally, in modern style, logical consequence is specified in a twofold way, namely syntactically and semantically. Its syntactic specification consists in presenting a formal deductive system by a set of logical axioms and a set of rules of deduction. Such a system may look as follows. Given a set Σ of statements and let α be some statement. Then we say that α is a logical consequence of Σ, symbolically $\Sigma \vdash \alpha$, if α can be *proved* from Σ. We then have to say what 'proved from Σ' means. Essentially, the idea is this. Assume we have certain purely logical axioms, logical truths so to speak, which can be used freely in any proof. Moreover, consider the statements in Σ as given assumptions that can equally be used freely in the proof. If we can then 'derive' α from these given statements using the rules of deduction we say that α can be proved from Σ. A deductive system of this sort is called a *Hilbert syle deductive system.* There are various other types of deductive systems such as those introduced by Gentzen or natural deduction type systems introduced by Prawitz. There are also logical systems which put restrictions on the use of axoims and assumptions in proofs such as resource logics in which assumptions are viewed as sort of resource that can be 'used up' in the course of the proof and can therefore not be used freely in the proof. In this introductory chapter to logic we will present a Hilbert style deductive system that which goes back to Hilbert and Bernays [30]. The reader

should note that a proof will be a completely *formal* procedure involving just the formal manipulation of symbols.

The semantic approach to the concept of logical consequence invokes the notion of truth. As we said, from the semantic point of view, to say that α is a logical consequence of Σ means that α is true whenever all statements of Σ are true, in symbols $\Sigma \models \alpha$. In this, clearly, two things must be made precise. First we need to make precise what it means to say that a statement is true. Second, we need to make precise what it means to say "whenever α is true" We will see shortly how this works.

Once we have defined logical consequence in this twofold way, there arises a problem. We need to study how these two notions of logical consequence, syntactic and semantic, are related. More precisely, we want to prove *soundness* of the logic. This means we need to show that $\alpha \vdash \beta$ implies $\alpha \models \beta$. This is a natural requirement saying that syntactic consequence implies semantic consequence. If we can prove the other direction too, i.e. that semantic consequence implies syntactic consequence, we say that the logical system is *complete*. Generally a logical system is required to be sound. There are well established logics, however, which are not complete.

In this chapter we present the basics of *classical propositional logic* in the style described. In particular, we will see that classical propositional logic is sound and complete.

2.1.2 Syntax of Classical Propositional Logic

We start by defining the *language of classical propositional logic*. We are aware of the fact that the way we do this does not meet the standards of linguistic precision the pure logician might expect.

To those readers who are interested in a presentation of the highest standards we recommend Friedrichdorf's excellent textbook [16].

The language of propositional logic is built up from the following symbols:
1) A set of propositional variables
2) Symbols for the connectives: $\neg$ (negation)and $\rightarrow$ (implication)
3) Brackets

We define the set Fml of well formed formulas of the language of propositional logic, formulas for short, inductively by the following clauses:

- Every propositional variable p is a formula.
- If α and β are formulas, so are $\neg\alpha$ and $(\alpha \rightarrow \beta)$.

If there is no danger of misunderstandings we omit the brackets. To be a bit more precise, Fml is the smallest set satisfying the above conditions.

Fix a certain variable p and define the symbols $\top$ and $\bot$ as abbreviations for $p \rightarrow p$ and $\neg\top$ respectively. Moreover we use the following abbreviations.

- $\alpha \wedge \beta$ for $\neg(\alpha \rightarrow \neg\beta)$
- $\alpha \vee \beta$ for $(\neg\alpha \rightarrow \beta)$

- $\alpha \leftrightarrow \beta$ for $(\alpha \to \beta) \wedge (\beta \to \alpha)$

2.1.3 A Hilbert style deductive system for classical propositional logic

Definition 2.1 *A (Hilbert style) deductive system consists of a set of formulas called axioms and a set of rules of inference. Given a deductive system L and a set Σ of formulas. A proof from Σ in L is a sequence of formulas such that each element is either an axiom or an element of Σ or it can be inferred from previous elements using a rule of inference. The elements of Σ are called assumptions. If α is the last element of the sequence, the sequence is called a proof of α from Σ. We say that α is provable from Σ, denoted by $\Sigma \vdash \alpha$, if there exists a proof of α from Σ. If Σ is a set of axioms, we write $\vdash \alpha$.*

Definition 2.2 *$\mathcal{H}$ is a deductive system with four axiom schemes and one rule of inference. More precisely, for any formulas α, β, γ, the following formulas are axioms:*

- *Axiom scheme* 1: $(\alpha \to (\beta \to \alpha))$
- *Axiom scheme* 2: $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
- *Axiom scheme* 3: $(\alpha \to (\neg\alpha \to \beta)$
- *Axiom scheme* 4: $(\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta)$

The rule of inference is called modus ponens (MP for short). For any formulas α, β: if $\vdash \alpha$ and $\vdash \alpha \to \beta$, then $\vdash \beta$.

$$\frac{\vdash \alpha, \vdash \alpha \to \beta}{\beta}$$

The proof of the following important lemma is an exercise in deduction in the above deductive system.

Lemma 2.1 *Far any δ we have $\vdash \delta \to \delta$.*

Proof. Consider axiom scheme $A2$ for $\alpha =: \delta$ and $\beta =: \delta \to \delta$ and $\gamma =: \delta$. Then we have by $A2$

$$(1) \vdash (\delta \to ((\delta \to \delta) \to \delta)) \to (\delta \to (\delta \to \delta)) \to (\delta \to \delta))$$

Consider axiom $A1$ for $\alpha =: \delta$ and $\beta =: \delta \to \delta$. Then we have by $A1$

$$(2) \vdash (\delta \to ((\delta \to \delta) \to \delta))$$

Modus ponens applied to (1) and (2) gives us

$$(3) \vdash \delta \to ((\delta \to \delta) \to \delta) \to \delta)$$

By $A1$ we have

$$(4) \vdash \delta \to (\delta \to \delta)$$

Modus ponens aplied to (4) and (5) yields what we want.

$$(5) \vdash \delta \to \delta$$

■

We have the following derived rules

$R1$:

$$\frac{\Sigma \vdash \alpha}{\Sigma \vdash \beta \to \alpha}$$

$R2$:

$$\frac{\Sigma \vdash \alpha \to (\beta \to \gamma),\, \Sigma \vdash \alpha \to \beta}{\Sigma \vdash \alpha \to \gamma}$$

$R3$:

$$\frac{\Sigma \vdash \alpha,\, \Sigma \vdash \neg\alpha}{\Sigma \vdash \beta}$$

$R4$:

$$\frac{\Sigma \vdash \alpha \to \beta, \Sigma \vdash \neg\alpha \to \beta,}{\Sigma \vdash \beta}$$

What does 'derived rule' mean? Consider $R1$. It says that given a set Σ of assumptions and suppose $\Sigma \vdash \alpha$. Then $R1$ says that $\Sigma \vdash \beta \to \alpha$. In fact, suppose $\Sigma \vdash \alpha$. This says that there exists a proof of α from Σ: $...\alpha$. Considering that $\alpha \to (\beta \to \alpha)$ is an axiom and using modus ponens we see that $...\alpha$, $\alpha \to (\beta \to \alpha)$, $\beta \to \alpha$ is a proof of $\beta \to \alpha$ from Σ. As to $R2$ suppose that $...\alpha \to (\beta \to \gamma)$ and $...\alpha \to \beta$ are proofs from Σ. Using Axiom scheme 2 and aplying modus ponens twice we see that the following sequence is a proof from Σ: $...\alpha \to (\beta \to \gamma ...\alpha \to \beta$, $(\alpha \to (\beta \to \gamma) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$, $(\alpha \to \beta) \to (\alpha \to \gamma)$, $\alpha \to \gamma$.

Analogously we can prove that $R3$ and $R4$ are in fact derived rules.

DO THE PROOFS.

The following lemma states some obvious facts about provability.

Lemma 2.2

- *(i) If* $\Sigma \subset \Omega$ *and* $\Sigma \vdash \alpha$*, then* $\Omega \vdash \alpha$
- *(ii) Suppose* $\Sigma \vdash \delta$ *for all* $\delta \in \Delta$ *and* $\Delta \vdash \alpha$*. Then* $\Sigma \vdash \alpha$
- *(iii) Suppose* $\Sigma \vdash \alpha$ *and* $\Sigma \vdash \alpha \to \beta$*. Then* $\Sigma \vdash \beta$
- *(iv)* $\Sigma \vdash \alpha$ *iff there exists a finite* $\Delta \subset \Sigma$ *such that* $\Delta \vdash \alpha$

Definition 2.3 *We call a set of formulas* Σ *consistent if not* $\Sigma \vdash \bot$*. We say* Σ *is maximal consistent if it is consistent and does not admit a proper consistent extension.*

Lemma 2.3 *A set of formulas* Σ *is consistent iff every finite subset of* Σ *is consistent.*

Proof. For the direction from left to right suppose Σ is consistent and there exists a finite $\Delta \subset \Sigma$ which is inconsistent. This means that $\Delta \vdash \bot$. By lemma 2.2 we then have $\Sigma \vdash \bot$. Thus Σ would be inconsistent contrary to the hypothesis.

For the other direction assume every finite $\Delta \subset \Sigma$ is consistent and Σ is inconsistent. It follows that $\Sigma \vdash \bot$. By lemma 2.2 there exists a finite $\Delta \subset \Sigma$ such that $\Delta \vdash \bot$. But this would mean that Δ is inconsistent contrary to the hypothesis. ■

The next lemma is called *Lindenbaum's Lemma*

Lemma 2.4 *Let Σ be consistent. Then there exists a maximal consistent set Ω such that $\Sigma \subset \Omega$*

In the sequel we assume the language to be denumerable although, later on in the book, we also want to admit non-denumerable languages. All theorems proved in this chapter also hold for non-denumerable languages. In some of the proofs we then have to use Zorn's lemma.

Proof. Choose an enumeration $\alpha_0, \alpha_1, \alpha_2...$ of all formulas.

Then define a sequence of sets of formulas $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2...$ as follows: $\Sigma_0 = \Sigma$, $\Sigma_{n+1} =: \Sigma_n \cup \{\alpha_n\}$, if $\Sigma \cup \{\alpha_n\}$ is consistent, otherwise $\Sigma_{n+1} =: \Sigma_n$ Let Ω be the union of all these sets, i.e. $\Omega = \bigcup \Sigma_n$

First note that Ω is an extension of Σ by construction.

We claim that Ω is maximal consistent. To see this, first note that any Σ_n is consistent by construction. Assume Ω is inconsistent. This would by **??** mean that a there exists a finite $\Delta \subset \Omega$ that is inconsistent. But we have $\Delta \subset \Sigma_n$ for some n. Thus Σ_n would be inconsistent contrary to the way it was constructed. It follows that Ω is consistent.

We still need to prove that Ω is maximal consistent. Assume it is not maximal consistent. This means that there exists a proper consistent extension Ω^* of Ω which is consistent. Then there exists a formula $\alpha \in \Omega^*$ such that not $\alpha \in \Omega$. We have $\alpha = \alpha_n$ for some n in the above enumeration. Since not $\alpha_n \in \Omega$, we have that in the above construction not $\alpha \in \Sigma_{n+1}$. But this means that $\Sigma_n \cup \{\alpha\}$ is inconsistent and thus Ω^* is inconsistent because $\Sigma_n \cup \{\alpha\} \subset \Omega^*$, which is a contradiction. ■

The next theorem is called the *Deduction Theorem.*

Theorem 2.1 $\Sigma \cup \{\alpha\} \vdash \beta$ *iff* $\Sigma \vdash \alpha \rightarrow \beta$

Proof. For the direction from right to left assume that there exists a proof of $\alpha \rightarrow \beta$ from Σ. Then by modus ponens there exists a proof of β from $\Sigma \cup \{\alpha\}$, which means that $\Sigma \cup \{\alpha\} \vdash \beta$.

The direction from left to right is less obvious. The proof of this direction is by induction on the length of the proof of β from $\Sigma \cup \{\alpha\}$. More precisely, we prove the following. Let $\delta_1, \delta_2, ..., \delta_n = \beta$ be a proof of β from $\Sigma \cup \{\alpha\}$. Then we

have by the induction hypothesis that $\Sigma \vdash \alpha \to \delta_1, \Sigma \vdash \alpha \to \delta_2, ...\Sigma \vdash \alpha \to \delta_{n-1}$ We then need to show that $\Sigma \vdash \alpha \to \delta_n$. Consider two cases.

Case 1: $\delta_j \in \Sigma$ or δ_j is an axiom, then $\Sigma \vdash \delta_j$. Applying the derived rule $R1$ we get $\Sigma \vdash \alpha \to \beta$.

Case 2: δ_j is obtained by modus ponens from two preceding formulas, i.e. there exist $j_1 \in \{1, ..., n-1\}$ such that δ_j is obtained from δ_{j_1} and $\delta_{j_1} \to \delta_j$ by modus ponens. By the induction hypothesis we have $\Sigma \vdash \alpha \to \delta_{j_1}$ and $\Sigma \vdash \alpha \to \delta_{j_1} \to \delta_j$. Applying the derived rule R2 gives us $\Sigma \vdash \alpha \to \beta$ ■

Suppose $(\alpha \to \beta) \in \Omega$ and $\alpha \in \Omega$. We prove that $\alpha \in \Omega$. By $i)$ we have $\Omega \vdash \alpha$ and $\Omega \vdash \alpha \to \beta$. By modus ponens we have $\Omega \vdash \beta$ and thus by (i) $\beta \in \Omega$.

For the other direction it suffices to show that if not $\alpha \in \Omega$, then $(\alpha \to \beta) \in \Omega$ and if $\beta \in \Omega$, then $\alpha \to \beta) \in \Omega$.

So assume that not $\alpha \in \Omega$. By (ii) we have $\neg\alpha \in \Omega$ and thus $\Omega \vdash \neg\alpha$. Using the derived rule $R1$ we then get $\Omega \vdash \alpha \to \neg\alpha$. On the other hand we have by axiom 3 $\Omega \vdash \alpha \to (\neg\alpha \to \beta$. Applying the derived rule $R2$ we get $\Omega \vdash \alpha \to \beta$ and thus $(\alpha \to \beta) \in \Omega$.

Finally assume that $\beta \in \Omega$. By the derived rule $R1$ we have $\Omega \vdash \alpha \to \beta$ and thus $(\alpha \to \beta) \in \Omega$

Lemma 2.5 *If not $\Sigma \vdash \alpha$, then $\Sigma \cup \{\neg\alpha\}$ is consistent.*

Proof. Given the hypothesis, we need to prove that not $(\Sigma \cup \{\neg\alpha\} \vdash \bot$. For this we show that $\Sigma \cup \{\neg\alpha\} \vdash \bot$ implies $\Sigma \vdash \alpha$. So let

$$(1)\ \Sigma \cup \{\neg\alpha\} \vdash \bot$$

We have by lemma 2.1 and the detinition of the symbol $\top$

$$(2)\ \Sigma \cup \{\neg\alpha\} \vdash \top$$

We get by (1) and (2) and the derived rule $R3$ that

$$(3)\ \Sigma \cup \{\neg\alpha\} \vdash \alpha$$

It follows from (3) and the Deduction Theorem that

$$4\ \Sigma \vdash \neg\alpha \to \alpha$$

Again by lemma 2.1 we have

$$5\ \Sigma \vdash \alpha \to \alpha$$

The derived rule $R4$ applied to (4) and (5) then gives us

$$\Sigma \vdash \alpha$$

■

Definition 2.4 *let Σ be a consistent set. Then call Σ a theory or synonymously deductively closed if $\Sigma \vdash \alpha$ implies $\alpha \in \Sigma$. Call Σ complete if for any α we have either $\alpha \in \Sigma$ or $\neg\alpha \in \Sigma$.*

Theorem 2.2 *Let Σ be consistent. Then the following statements are equivalent*

- *(i) Ω is maximal consistent.*
- *(ii) Ω deductively closed, i.e. a theory.*
- *(iii) Ω is complete.*

Proof. We prove that (i) implies (ii). So let Ω be maximal consistent. Consider the set $\Delta := \{\beta \mid \Omega \vdash \beta\}$. We have $\Omega \subset \Delta$. Moreover, Δ is consistent because $\Delta \vdash \bot$ would, by 2.2, imply $\Omega \vdash \bot$ contradicting the consistency of Ω. So Δ is a consistent extension of Ω. Since Ω is maximal consistent, it follows that $\Delta = \Omega$. We have proved that $\Omega \vdash \alpha$ implies $\alpha \in \Omega$ which means that Ω is deductively closed.

We now prove that (ii) implies (iii).

Assume that both α and $\neg\alpha$ are in Ω. Then (i) would give us $\Omega \vdash \alpha$ and $\Omega \vdash \neg\alpha$ and, by the derived rule $R3$ we would have $\Omega \vdash \bot$, which is a contradiction because Ω is assumed to be consistent. It follows that at most one of the formulas α and $\neg\alpha$ are in Ω. Assume now that neither α nor $\neg\alpha$ is in Ω. This would mean, since Ω is maximal consistent, that the sets $\Omega \cup \{\alpha\}$ and $\Omega \cup \{\neg\alpha\}$ are inconsistent, i.e. $\Omega \cup \{\alpha\} \vdash \bot$ and $\Omega \cup \{\neg\alpha\} \vdash \bot$. It would then follow by the Deduction Theorem that $\Omega \vdash \alpha \to \bot$ and $\Omega \vdash \neg\alpha \to \bot$. By the derived rule $R4$ we would have $\Omega \vdash \bot$, a contradiction.

We still need to prove that (iii) implies (i). Supose Ω is complete. Assume it's not maximal consistent. This means that there exists a consistent extension Ω^* of Ω a formula α such that $\alpha \in \Omega^*$ and not $\alpha \in \Omega$. Since Ω is complete we have $\neg\alpha \in \Omega$ and thus $\neg\alpha \in \Omega^*$. Again using the derived rule $R3$ would give us $\Omega^* \vdash \bot$ contradicting the consistency of Ω^*. It follows that Ω does not admit a proper consistent extension, i.e. that it is maximal consistent.

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Theorem 2.3 *Let Ω be maximal consistent. Then we have $(\alpha \to \beta) \in \Omega$ iff not $\alpha \in \Omega$ or $\beta \in \Omega$*

Proof. Suppose $(\alpha \to \beta) \in \Omega$ and $\alpha \in \Omega$. We prove that $\beta \in \Omega$. We have $\Omega \vdash \alpha$ and $\Omega \vdash \alpha \to \beta$. By modus ponens we have $\Omega \vdash \beta$ and, since Ω is deductively closed, $\beta \in \Omega$.

For the other direction we need to show that if not $\alpha \in \Omega$, then $(\alpha \to \beta) \in \Omega$, and if $\beta \in \Omega$, then $\alpha \to \beta) \in \Omega$.

So assume that not $\alpha \in \Omega$. By 2.2 we have $\neg\alpha \in \Omega$ and thus $\Omega \vdash \neg\alpha$. Using the derived rule $R1$ we then get $\Omega \vdash \alpha \to \neg\alpha$. On the other hand we have by $A3$ $\Omega \vdash \alpha \to (\neg\alpha \to \beta$. Applying the derived rule $R2$ we get $\Omega \vdash \alpha \to \beta$ and thus, since Ω is deductively closed, $(\alpha \to \beta) \in \Omega$.

Finally assume that $\beta \in \Omega$. By the derived rule $R1$ we have $\Omega \vdash \alpha \to \beta$ and thus $(\alpha \to \beta) \in \Omega$. ■

2.1.4 Semantics of Classical Propositional Logic

As indicated earlier, we now approach the concept of logical consequence from the semantic point of view. For this we need, as already mentioned, a theory of truth. In the following definition we explain what it means for a formula α to be true under a valuation V.

Definition 2.5 *A valuation V is any function assigning a truth value to any formula, i.e.* 1 *or* 0 *such that*

- $(V)(\neg\alpha) = 1$ *iff* $V(\alpha) = 0$
- $V(\alpha \to \beta) = 1$ *iff* $V(\alpha) = 0$ *or* $V(\beta) = 1$

We say that α is true under V if $V(\alpha) = 1$. If α is true under any valuation V we say that α is a (classical) tautology.

Given a set Σ of formulas and a valuation V such that $V(\alpha) = 1$ for all $\alpha \in \Sigma$ we say that V is a model for Σ.

In the above definition we used the "or" (disjunction) of the English language. In the meta language, i.e. in English, a we define the disjunction to be true if at least one disjunct is true.

The reader may think of a valuation V as follows. Given any function $V : Var \to \{0, 1\}$. Then V can be extended in a unique way to a valuation (again denoted) by V. This means that it suffices to specify a valuation by specifying its values for the propositional variables.

In the next lemma we again use a meta connective, namely the "and" (conjunction) of English. In the meta language (English) we define a conjunction to be true iff both conjuncts are true.

Lemma 2.6 *For any valuation V we have*

- $V(\top) = 1$
- $V(\bot) = 0$
- $V(\alpha \wedge \beta) = 1$ *iff* $V(\alpha) = 1$ *and* $V(\beta) = 1$

2.1.5 Soundness and Completeness

Lemma 2.7 *Let Σ be maximal consistent. Σ has a model.*

Proof. We define V_Σ as follows. If $p \in \Sigma$ then $V_\Sigma(p) = 1$ otherwise $V_\Sigma(p) = 0$ We prove that V_Σ has the properties above by induction on the construction of wff.

If α is a variable the claim holds by the definition of V_Σ.

In the case $\alpha = \neg\beta$ we argue as follows. $V_\Sigma(\alpha) = 1$ iff $V_\Sigma(\beta) = 0$ This is, by the induction hypothesis, the case iff not $\beta \in \Sigma$ and, since Σ is maximal consistent, this is equivalent to $\alpha \in \Sigma$. The other cases are analogous. ■

The following theorem is an immediate consequence of the above lemma.

Theorem 2.4 *Any consistent set Σ of wff has a model, i.e there exists a valuation V such that $V(\alpha) = 1$ for all $\alpha \in \Sigma$.*

We now give the semantic definition of logical consequence.

Definition 2.6 *Given a set Σ of formulas and a formula α. We say that α is a semantic consequence of Σ, in symbols $\Sigma \models \alpha$, if for model V of Σ we have $V(\alpha) = 1$*

It is routine to verify the following. Given any axiom α, then we have for any valuation V that $V(\alpha) = 1$, i.e. every axiom is a tautology. Moreover, given two formulas α and β and a valuation V such that $V(\alpha) = 1$ and $V(\alpha \to \beta) = 1$, then we have $V(\beta) = 1$. Thus any axiom is true under any valuation and modus ponens preserves truth.

The next theorem, which expresses *soundness* of classial propositional logic, is an immediate consequence of the above facts.

Theorem 2.5 $\Sigma \vdash \alpha$ *implies* $\Sigma \models \alpha$

The following theorem is the *Completeness Theorem* of classical propositional logic. As always in logic completeness is more hard to prove than soundness.

Theorem 2.6 $\Sigma \models \alpha$ *implies* $\Sigma \vdash \alpha$

Proof. We show that not $\Sigma \vdash \alpha$ implies that not $\Sigma \models \alpha$. Assume that not $\Sigma \vdash \alpha$. It follows by lemma 2.5 that $\Gamma =: \Sigma \cup \{\neg\alpha\}$ is consistent. By theorem 2.4 Γ has a model, i.e. there exists a valuation V such that $V(\beta) = 1$ for all $\beta \in \Gamma$. In particular we then have $V(\beta) = 1$ for all $\beta \in \Sigma$ and $V(\neg\alpha) = 1$ and thus $V(\alpha) = 0$. But this means that not $\Sigma \models \alpha$

■

2.1.6 Compactness

Suppose that $\Sigma \vdash \alpha$. Since any proof of α from Σ involves only finitely many assumptions there exists a finite $\Delta \subset \Sigma$ such that $\Delta \vdash \alpha$.

We therefore have the following theorem, which is known as the *Compactness Theorem* of classical propositional logic.

Theorem 2.7 $\Sigma \vdash \alpha$ *(or equivalently* $\Sigma \models \alpha$*) iff there exists a finite* $\Delta \subset \Sigma$ *such that* $\Delta \models \alpha$ *or (equivalently* $\Delta \vdash \alpha$*).*

2.1.7 Lattices

We define the concepts of
a lattice
an orthocomplemented lattice
Boolean algebra

Definition 2.7 *A partially ordered set (in short poset) is a pair $\langle L, \leq \rangle$, where L is a non empty set and $\leq$ is a binary relation satisfying the following conditions*

- *(i) $A \leq A$ for any $A \in L$ (reflexivity)*
- *(ii) If $A \leq B$ and $B \leq A$, then $A = B$ (antisymmetry)*
- *If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity)*

Call $\leq$ a partial order.

Let $S \subset L$. An upper bound of S is an element $a \in L$ such that $b \leq a$ for all $b \in S$. a least upper bound of S is is an element $a \in L$ such that a is an upper bound and $a \leq b$ for every upper bound of S. Analogously we define the concept of lower bound of S and the concept of a greatest lower bound.

It is readily seen that, if a least upper bound exists for S, then it is unique and analogously for the greatest lower bound.

Definition 2.8 *The partially ordered set $\langle L, \leq \rangle$ is called a lattice if for any two elements A and B there exists the least upper bound denoted by $A \vee B$ and the greatest lower bound denoted by $A \wedge B$ and there exists a zero elment 0 and a unit element 1, i.e. elements such that for all $A \in L$ we have $0 \leq A$ and $A \leq 1$. The lattice is called complete if any subset of L has a greatest lower bound and a smallest upper bound.*

The reader may realise that we denote the least upper bound (greatest lower bound) in a lattice by the same symbol as the propositional connectives of conjunction (disjunction) which should not lead to any confusion. It is readily verified that greatest lower bounds and least uper bounds are uniquely determined if they exist. The same is true for the zero and unit element.

Definition 2.9 *We call a lattice distributive if the following holds for any $A, B, C \in L$*

$$A \vee (B \wedge C) = (A \vee B \wedge (A \vee C)$$

Definition 2.10 *Let $\mathcal{L}$ be a lattice. A map*

$$A \mapsto A^{\perp}$$

is called an orthocomplementation and $A^{\perp}$ the orthocomplement of A if it has the following properties

- *(i)* $A^{\perp^{\perp}} = A$
- *(ii) if* $A \leq B$ *then* $B^{\perp} \leq A^{\perp}$
- *(iii)* $A \wedge A^{\perp} = 0$
- *(iv)* $A \vee A^{\perp} = 1$
  We call a $\langle L, \leq, ^{\perp} \rangle$ *an orthocomplemented lattice or simply an ortholattice if* $^{\perp}$ *is an orthocomplement of the lattice* $\langle \leq \rangle$.

Definition 2.11 *A Boolean algebra is an orthocomplemented and distributive lattice.*

2.1.8 The Lindenbaum algebra

We now make the connection between classical logic on the hand and certain algebraic structures on the other. In the case of classical logic it is the concept of a Boolean algebra that constitutes its algebraic counterpart. At this stage we introduce the concept of a Lindenbaum algebra in the traditional manner. *We will look at this concept in a new way in Chapter* 9

Given a consistent set Σ. We then write $\vdash_{\Sigma} \alpha$ for $\Sigma \vdash \alpha$. Call two formulas α and β Σ-equivalent, in symbols $\equiv_{\Sigma}$, if $\vdash_{\Sigma} \alpha \leftrightarrow \beta$. We will see that this is in fact an equivalence relation. Denote the equivalence class of α by $[\alpha]$ and denote the set of these equivalence classes by A_{Σ}. These equivalence classes form a (Boolean) algebra in a natural way. Namely, define $[\alpha] \leq_{\Sigma} [\beta]$ if $\Sigma \cup \{\alpha\} \vdash \beta$ This is well defined as we will see. Similarly define $[\alpha]^{*}=: [\neg\alpha]$. Again, this is well defined. The proof of the following theorem is straightforward and is, in other books sometimes left to the reader as an exercise. We present it here in more detail than is usual because, in this book, we will look at the concept of a Lindenbaum algebra from a more general point of view later on will prove a more general theorem of which the next theorem is a special case. The reader can then compare the difference in level of these approaches. The gist of this viewpoint is that we can view the Lindenbaum algebra as an operator algebra in a natural way. This viewpoint will permit us to generalise the concept of a Lindenbaum algebra and to prove a more general theorem of which the next theorem is a special case.

Theorem 2.8 $LT_{\Sigma} = \langle A_{\Sigma}, \leq_{\Sigma}, * \rangle$ *is a Boolean algebra.*

The proof of the above theoem relies on certain facts of classical logic which we stated in the following lemmata. It is routine, and we therefore do not present it in all details. Rather we describe the general procedure of the proof elaborating on just a few typical items. The reader is invited to work out the full proof as an exercise.

It should be noted that theorem is a purely syntactic statement and so are the lemmata below. The reader might therefore expect purely syntactic proofs. In fact, this way of proceeding is perfectly feasible. It would, however, as the

reader can easily convince himself, be fairly tedious at least compared to the way we will actually proceed.

Rather we will prove the syntactic lemmata below in a more transparent way semantically. What does this mean? It means that by the soundness and completeness theorems all the syntactic statements involved in the lemmata below are equivalent to certain semantic statements. It therefore suffices -by soundness and completeness- to prove the semantic equivalents of the lemmata below. All the statements to be proved reduce to proving statements of the form $\vdash_\Sigma \alpha$. By the soundness and completeness we know that these statements are equivalent to statements of the form $\Sigma \models \alpha$. In order to prove such a statement we can, by the semantic definition of logical consequence, proceed as follows. Given any valuation V such that $V(\varphi) = 1$ for all $\varphi \in \Sigma$, i.e. V is a model of Σ, then show that $V(\alpha) = 1$.

In the sequel Σ denotes a consistent set of formulas.

Lemma 2.8
- (1) $\vdash_\Sigma \alpha \leftrightarrow \alpha$
- (2) *If* $\vdash_\Sigma \alpha \leftrightarrow \beta$, *then* $\vdash_\Sigma \beta \leftrightarrow \alpha$
- (3) *If* $\vdash_\Sigma \alpha \leftrightarrow \beta$ *and* $\vdash_\Sigma \beta \leftrightarrow \gamma$, *then* $\vdash_\Sigma \alpha \leftrightarrow \gamma$

Proof. We restrict ourselves to (3). The other cases are proved analogously. Assume that $\Sigma \models \alpha \leftrightarrow \beta$ and $\Sigma \models \beta \leftrightarrow \gamma$. We need to show that $\Sigma \models \alpha \leftrightarrow \gamma$. Let V be any model of Σ. Then we have by the first hypothesis that either $V(\alpha) = V(\beta) = 1$ or $V(\alpha) = V(\beta) = 0$. In the first case we get from the second hypothesis that $V(\beta) = V(\gamma) = 1$. It follows that $V(\alpha \leftrightarrow \beta) = 1$. In the second case we get from the second hypothesis that $V(\alpha) = V(\gamma) = 0$ and thus $V(\alpha \leftrightarrow \gamma) = 1$. ■

Lemma 2.9 *Suppose that* $\vdash_\Sigma \alpha \leftrightarrow \alpha'$ *and* $\vdash_\Sigma \beta \leftrightarrow \beta'$. *Then we have*

- (1) $\vdash_\Sigma \alpha \wedge \beta \leftrightarrow \alpha' \wedge \beta'$
- (2) $\vdash_\Sigma \alpha \vee \beta \leftrightarrow \alpha' \vee \beta'$
- (3) $\vdash_\Sigma \neg\alpha \leftrightarrow \neg\alpha'$
- (4) $\vdash_\Sigma (\alpha \rightarrow \beta) \leftrightarrow (\alpha' \rightarrow \beta')$

Proof.

■

Lemma 2.10
- (1) $\vdash_\Sigma \alpha \leftrightarrow \alpha$
- (2) *If* $\vdash_\Sigma \alpha \rightarrow \beta$ *and* $\vdash_\Sigma \beta \rightarrow \alpha$, *then* $\vdash_\Sigma \alpha \leftrightarrow \beta$.
- (3) *If* $\vdash_\Sigma \alpha \rightarrow \beta$ *and* $\beta \rightarrow \gamma$, *then* $\vdash_\Sigma \alpha \rightarrow \gamma$

The leave the proof of the following lemmata to the reader as an exercise in the procedure applied above.

Lemma 2.11
- (1) $\vdash_\Sigma (\alpha \wedge \beta) \to \alpha$
- (2) $\vdash_\Sigma (\alpha \wedge \beta) \to \beta$
- (3) *If* $\vdash_\Sigma \gamma \to \alpha$ *and* $\vdash_\Sigma \gamma \to \beta$*, then* $\vdash_\Sigma \gamma \to (\alpha \wedge \beta)$*.*
- (4) $\vdash_\Sigma \alpha \to \alpha \wedge \beta$
- (5) $\vdash_\Sigma \beta \to \alpha \wedge \beta$
- (6) $\vdash_\Sigma \bot \to \alpha$
- (7) $\vdash_\Sigma \alpha \top$

Lemma 2.12 *If* $\vdash_\Sigma \gamma \to \alpha$ *and* $\vdash_\Sigma \gamma \to \beta$*, then* $\vdash_\Sigma \gamma \to \alpha \wedge \beta$*.*

Lemma 2.13
- (1) $\vdash_\Sigma \alpha \leftrightarrow \neg\neg\alpha$
- (2) $\vdash_\Sigma \alpha \wedge \neg\alpha \leftrightarrow \bot$
- (3) $\vdash_\Sigma \alpha \vee \neg\alpha \leftrightarrow \top$
- (4) $\alpha \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

We now prove the theorem.

Proof. We first need to show that $\equiv_\Sigma$ is in fact an equivalence relation. We just verify transitivity. The other conditions are proved analogously. So let $\alpha \equiv_\Sigma \beta$ and $\beta \equiv_\Sigma \gamma$. By the definition of $\equiv_\Sigma$ this says that $\vdash_\Sigma \alpha \leftrightarrow \beta$ and $\vdash_\Sigma \beta\gamma$. Lemma 2.8 then gives us $\alpha \equiv_\Sigma \gamma$.

We now need to see that $\leq$ is well defined. For this we must verify that given $\alpha \equiv_\Sigma \alpha'$ and $\beta \equiv_\Sigma \beta'$, $\vdash_\Sigma \alpha \to \beta$ implies $\vdash \alpha' \to \beta'$. But this is lemma 2.9 (4). That $\leq$ is a partial order follows from 2.10.

$[\bot]$ is the smallest element. In fact $\bot \to \alpha$ is a tautology for any α. From the fact $\alpha \to \top$ is a tautology it follows that $[\top]$ is the greatest element.

For given α and β it follows from 2.11 that $[\alpha \wedge \beta]$ is the greatest lower bound of $[\alpha]$ and $[\beta]$.

Finally lemma

■

2.2 Basics of Nonmonotonic Logic

2.2.1 What is nonmonotonic logic?

Make some remarks about the nature of nonmonotonic logic. Start from autoepistemic logic. Classical logic is monotonic. Given a set Σ of assumptions and a formula α such that $\Sigma \vdash \alpha$. If we add more assumptions to Σ so as to get Σ^* we will still have $\Sigma^* \vdash \alpha$. 'More information' cannot invalidate inferences drawn on the basis of 'less information'. This is what monotonicity

means. In recent decades, logicians have studied modes of reasoning that do not have this property. In these so-called *non-monotonic logics* 'old inferences' may be invalidated by 'new information'. What reason can there be for this phenomenon? One reason may be incomplete information. This is for instance the case in commonsense reasoning. If we view a common sense reasoner's activity as 'jumping to conclusions' on the basis of certain 'pieces of information', it seems quite natural that certain of his conclusions cannot be maintained in the light of additional information.

Another source of non-monotonicity is perfect introspection of the (reasoning) agent. Imagine a reasoner, i.e. an agent who can infer propositions from sets of assumptions. Suppose, moreover, this reasoner has an additional ability. Namely assume that whenever he can, in his system of reasoning, infer a certain proposition α from a certain set Σ of assumptions, he can, in the same system, infer the proposition saying "I can infer α" denoted by $I\alpha$ and whenever he cannot infer α from Σ he can infer the proposition "I cannot infer α", i.e. $\neg I\alpha$. The former ability is called positive introspection, the latter is called negative introspection. Assume a consistent agent having both abilities. We give a (still slightly informal) argument to the effect that such a reasoner cannot be monotonic. So assume he is monotonic. Given a set Σ of assumtions and let α be a proposition the reasoner cannot infer from Σ). By negative introspection he can then infer $\neg I\alpha$. Aassume that α is consistent with Σ and can be consistently added to Σ. Then, since he is assumed to be monotonic, he *can* infer α from α from $\Sigma \cup \{\alpha\}$ and thus by (positive) introspection he can infer $I\alpha$ from the enlarged set of assumtions. Since he is assumed to be monotonic, he can still infer $\neg I\alpha$. But this would mean that he is inconsistent. It follows that he cannot be monotonic.

The branch of non-monotonic logic that takes its origin in considerations of the above sort is called *autoepistemic logic*, see for instance [1] or [48]. We will come back to this in Chapter 9.

2.2.2 Non-monotonicity in quantum mechanics

The reader may, at this point, ask the question why we want to consider non-monotonic logics in our study of quantum logic. The answer is that non-monotonicity is, from the logical point of view, an essential feature of quantum mechanics. We encounter non-monotonic (logical) systems in nature so to speak. This has to do with Heisenberg's famous Uncertainty Principle, more generally the uncertainty relations we have in quantum mechanics. What are uncertainty relations? We cannot, at this stage, explain this quantitatively. But, qualitatively, it means the following. Consider, say, the electron of a hydrogen atom and assume a certain physical quantity of this electron, say its (total) energy E is measured. Through measurement we get a certain value, say μ. Viewing a measurement as a sort of proof we then have 'proved' the proposition $E = \mu$. Now assume we measure the position P of the electron. Again, we get a value, say λ, and we have proved the proposition $P = \lambda$. We are now, used to classical physics and classical logic as we are, inclined to say that we now know the

energy and the position of the electron and any subsequent measurement of the energy of the electron could only confirm the proposition $E = \mu$ and $P = \lambda$. It is an empirical and perhaps slightly surprising fact, however, that this is not the case. A subsequent measurement of the energy of the electron will even with certainty yield a value different from μ. The measurement of position invalidates the result of the measurement of energy. This is essentially what we mean by saying that there exists an uncertainty relation between energy and position. From the point of view of logic this is non-monotonicity.

2.2.3 Inference operations and consequence relations

Here we introduce the concept of an inference operation and as a special case that of a consequence relation. This is routine and still needs to be done.

We state some minimal condition a consequence relation is supposed to satisfy. The following are the minimal conditions as suggested by Gabbay in [**?**]. The reader may verify that the classical consequence relation $\vdash$ or equivalently $\models$ defined above in fact satisfies these conditions.

Reflexivity

$$\alpha \mid\!\sim \alpha$$

Cut

$$\frac{\alpha \wedge \beta \mid\!\sim \gamma,\ \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$

Restricted Monotonicity

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

In the paper by Kraus–Lehmann–Magidor (in short KLM, see [34], these three conditions are, as suggested by Gabbay in [**?**], considered to be the minimal conditions a respectable consequence relation should satisfy.
As observed in the KLM paper, any consequence relation satisfying the above conditions has the following property *AND*:

$$\frac{\alpha \mid\!\sim \beta, \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$$

For a given consequence relation $\mid\!\sim$ define

$$\alpha \equiv \beta \text{ iff } \alpha \mid\!\sim \beta \text{ and } \beta \mid\!\sim \alpha$$

2.2.4 The concept of a GKLM model

The semantics of consequence relations has its origin in investigations on the semantics of conditionals. The essential step in developing semantics of such a sort is the definition of a *model for a consequence relation* rather than a model for a formula. Having accomplished this is one of the merits of the seminal

paper by Kraus-Lehmann- Magidor [34]. In the sequel we give a definition due to Gabbay [18] which is a slight generalisation of the definition given in the KLM paper.

Definition 2.12 • *A Scott model for Fml is any function $s : Fml \to \{0,1\}$.*

- *A $GKLM$ (Generalised Kraus–Lehmann–Magidor) model is a structure of the form $\langle S, <, l\rangle$, where S is a non-empty set, $<$ is a binary relation on S and l is a function associating with each $t \in S$ a set of Scott models $l(t)$. The model is required to satisfy the smoothness condition stated in the next definition.*

Definition 2.13 *Let $\mathcal{M} = \langle S, <, l\rangle$ be a structure as described in the last definition. Let $t \in S$ and α a formula. Then define the satisfaction relation $t \models \alpha$ as follows:*

- *$t \models \alpha$ iff for all $s \in l(t)$ we have $s(\alpha) = 1$*
- *Let $A \subset S$. We say that t is $<$-minimal in A iff for all $t' \in A$ such that $t' < t$ we have $t' = t$. We say that A is smooth iff for every $t \in A$, either t is minimal in A or for some $s \in A$, $s < t$ and s is minimal in A.*
- *Let $[\alpha] = \{t \in S \mid t \models \alpha\}$. We say that $\mathcal{M}$ is smooth iff for all α, $[\alpha]$ is smooth.*
- *For a smooth model $\mathcal{M}$ we define the consequence relation $\mid\sim_{\mathcal{M}}$ as follows: $\alpha \mid\sim_{\mathcal{M}} \beta$ iff for all t minimal in $[\alpha]$, we have $t \models \beta$.*
- *Given a consequence relation $\mid\sim$ and a smooth model $\mathcal{M}$. We say $\mathcal{M}$ is a model for $\mid\sim$ iff $\mid\sim = \mid\sim_{\mathcal{M}}$.*

Chapter 3

Some Hilbert Space Theory

Abstract:. In order to understand the idea of quantum logic it is indispensable to have a basic knowledge of quantum mechanics and its mathematical formalism. Moreover, the reader needs such a basic knowledge in order to understand our interpretation of the Birkhoff-von Neumann paper, which initiated quantum logic. We therefore give an introduction to this in this chapter. We define the core concept of the mathematical formalism, namely the concept of a Hilbert space and develop elementary Hilbert space theory. In this we assume some familiarity with basic concepts of linear algebra on the part of the reader. In particular, we assume the reader to be familiar with the concept of a vector space, linear independence,... We also expect the reader to be familiar with elementary topological concepts such as metric space, continuity, convergence... We introduce and study the lattice of closed subspaces, equivalently the lattice of projections, of a Hilbert space, which constitutes the dominant algebraic structure in quantum logic. On this we will build heavily in later chapters. ■

3.1 The Concept of a Hilbert Space

In this chapter we first summarise some well known material from Hilbert space theory. In this we omit the proofs with few exceptions. The reader may find the proofs in most textbooks on Functional Analysis. We report most of the material without explicitly formulating definitions or theorems. It's only when we want to highlight certain concepts or results that we choose the explicit formulation as a definition or a theorem. Not all of the things we say about Hilbert spaces are actually needed in the subsequent chapters, but all of the results that are actually needed are included.

We also present an unpublished result by the authors on the characterisation of classical Hilbert lattices which will prove useful in our considerations later in the book.

The core mathematical structure of the formalism of quantum mechanics is that of a Hilbert space.

Definition 3.1 *Let H be a vector space over the real or the complex numbers or the quaternions and let $\| .. \|$ $H \to [, [0\infty)]$ be a function such that*

- $\| x \| = 0$ *iff* $x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Then we say that H is a normed space with norm $\|..\|$

Given a normed space H with norm $\|..\|$, we can define a metric d in a natural way, namely by $d(x, y) = \|x - y\|$. Thus a normed (linear) space is a metric space in a natural way.

We thus have the topological concepts of continuity, Cauchy-sequence... Let us just recall the definition of a Cauchy-sequence. Let M be a metric space and let $x_{nn\in N}$ be a sequence in M. We say that $x_{nn\in N}$ is a Cauchy-sequence if the following holds. For any $\epsilon > 0$ there exists an n_0 such that for any $n, m > n_0$ we have $d(x_n - x_m) < \epsilon$.

We call a metric space M *complete* if every Cauchy sequence in M converges. A complete normed space is called a *Banach space*.

Definition 3.2 *Let H be a vector space over scalar (skew) fields of the real or the complex numbers or the quaternions. A mapping $\langle ... \rangle \to$ from $H \times H$ into the scalar (skew) field is called an inner (scalar) product, if the following conditions are satisfied:*

- $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$
- $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- $\langle x, x\rangle \geq 0$
- $\langle x, x\rangle = 0$ *iff* $x = 0$ *if $\langle ... \rangle$ is an inner product we call $\langle H, \langle ... \rangle\ \rangle$ (or in abuse of notation just H) an inner product space (synonymously a Pre-Hilbert space).*

We get as a consequence that

$\langle x, y_1 + y_2\rangle = \langle x, y_1 + y_2\rangle\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$

The following is the Cauchy-Schwarz inequality. This important inequality will not be explicitly used in the sequel. We mention it because it basic in the sense that it plays an important part in the proofs of the theorems mentioned (not proved) in the sequel.

Let H be an inner product space. Then

- $|\ \langle x, y\rangle\ |^2 \leq \langle x, x\rangle\langle y, y\rangle$

we have equality iff x and y are linearly dependent.

We call two elements x and y of an inner product space *orthogonal* if $< x, y >= 0$. We write $x \perp y$.

Note that in inner product spaces we have the *Pythagorean theorem*: If x and y are orthogonal, then $\| x \|^2 + \| y \|^2 = \| x + y \|^2$.

Definition 3.3 *We call an inner product space a Hilbert space if it is complete as a normed space. Equivalently we may say that an inner product space is called a Hilbert space if it is a Banach space.*

The reader should note that we did not impose any condition on the dimension of a Hilbert space although, historically, the concept arose in connection with infinite dimensional vector spaces. In fact, the most interesting examples of Hilbert spaces in Functional Analysis are infinite-dimensional function spaces. We thus consider both finite-dimensional Hilbert spaces and infinite-dimensionl Hilbert spaces. There are, however, marked differences between the finite-dimensional and the infinite-dimensional case which should be kept in mind.

It is for instance true that any finite-dimensional inner product space is a Hilbert space, i.e. complete as a normed space. This does not hold in the infinite- dimensional case. There exist infinite-dimensional inner product spaces that are not complete. Other peculiarities of infinite-dimensional Hilbert spaces concern the role of subspaces and that of bases.

3.2 Closed subspaces and projections in Hilbert space

Given any vector space H and $S \subset H$. Then we say that S is a subspace of H (in the sense of Linear Algebra) if $0 \in S$ and if $x, y \in S$ then $\lambda x + \mu y \in S$ for any scalars λ and μ.

Given a finite-dimensional inner product space $\langle H, \langle\rangle\rangle$ and let S be a subspace of H. If we then restrict the inner product to S and denote its restriction again by $\langle\rangle$ then $\langle S, \langle\rangle\rangle$ is again a Hilbert space. We cannot expect this to hold in the infinite dimensional case because we cannot take for granted that $\langle S, \langle\rangle\rangle$ is complete. If, however, we require S to be closed then $\langle S, \langle\rangle$ is in fact a Hilbert space. It is the *closed subspaces* that in the infinite dimensional case play, essentially, the role of subspaces in the finite-dimensional case. Let us take a closer look at closed subspaces of a Hilbert space. To be precise, we call a subset S of a Hilbert space a closed subspace if S is a subspace in the sense of linear algebra and if S is a closed set in the norm topology.

For a given subset S of a Hilbert space we denote its closure, i.e. the smallest closed set containing S, by $\bar{S}$. It is easily seen that for any subspace S its closure $\bar{S}$ is a closed subspace. For two (not necessarily closed) subspaces A and B we denote by $A + B$ the smallest (not necessarily closed) subspace containing A and B. For two closed subspaces A and B denote by $A \vee B$ the smallest closed

subspace containing A and B. We clearly have $A + B \subset A \vee B$. We will see in that this may be be a proper inclusion. Call two subspaces A and B orthogonal if for any $x \in A$ and $y \in B$ we have $x \perp y$.

For any subspace A define its orthogonal complement $S^\perp$ by

$A^\perp =: \{x \in H \mid (\forall y \in A) x \perp y\}$ One can then prove that $\mathcal{A}$ is again a subspace. If A is closed so is $A = A^\perp$. Moreover, we have that A is a closed subspace iff $A = A^{\perp^\perp}$

Theorem 3.1 *Let A be a closed subspace of the Hilbert space H and $x \in H$. Then x has a unique decomposition $x = y + z$ with $y \in A$ and $z \in A^\perp$.*

Given a closed subspace A and $x \in H$. Let $x = y + z$ the unique decomposition of x as above. Then we call y the projection of x onto A. We denote, since there is no confusion, the mapping assigning to every $x \in H$ its projection onto A again by A. We call these mappings *projections.* Note that any projection A is *idempotent*, $A^2 = A$.

Theorem 3.2 (Projection Theorem) *Let H be a Hilbert Space, $x \in H$ and A be a closed subspace of H. Then there exists a unique $y \in A$ such that $d(x, y) = \inf_{z \in A} d(x, z)$ and we have $Ax = y$.*

3.3 Orthonormal systems and the Fourier expansion

Recall from linear algebra that any vector space and thus any Hilbert space - finite dimensional or not- admits a basis. A basis in the sense of linear algebra is a family of linearly independent vectors that spans the whole space. Every vector has then a unique representation as a linear combination of basis vectors. In the case of infinite-dimensional Hilbert spaces it is not the bases in the sense of linear algebra that play the dominant role but other systems, which are in general not bases in the sense of linear algebra. These systems are called *orthonormal bases.* Their main characteristic is that every vector has a (unique) 'Fourier expansion' in terms of such systems. In this subsection we summarise the main properties of orthonormal bases of a Hilbert space.

Definition 3.4 *Let $S \subset H$. We call S an orthonormal system if every $x \in S$ has norm 1, i.e. $\| x \| = 1$, and any two distinct elements x and y are orthogonal. We call an orthonormal system S an orthonormal basis if it is maximal in the sense that for any orthonormal system T such that $S \subset T$ we have $S = T$.*

Using Zorn's lemma one can prove that every Hilbert space and thus every closed subspace of a Hilbert space possesses an orthonormal basis. Moreover, it can be proved that any two orthonormal bases have the same cardinality. A Hilbert space having countable orthonormal bases is called separable.

The reader should note that, in the infinite-dimensional case, an orthonormal basis need not be a basis for the Hilbert space in the sense of linear algebra. It is a linearly independent set of elements which, however, need not (linearly) span the whole space. We will see shortly, however, that for any orthonormal basis S every vector of the Hilbert space has a 'Fourier expansion' in terms of S which in the finite-dimensional case reduces to a linear combination of the elements of S.

We also have, in the infinite-dimensional case, the following analogy with the finite-dimensional case. It is well known that given a finite-dimensional Hilbert space and any linearly independent family $x_1, ...x_n$ there exists an orthonormal set $\{y_1, ...y_n$ spanning the same space. Generally, given any countable linearly independent set of vectors $T = \{x_n, n \in N\}$ we may construct an orthonormal basis S such that the *closures* of the subspaces spanned by T nd S coincide. The reader may find this in the textbooks as the 'Gram-Schmidt construction'.

As already mentioned, the chief function of an orthonormal basis of Hilbert space H consists in the fact that any $x \in H$ has a Fourier expansion in terms of this orthonormal basis. We need, at this point, reflect on two things. First 'Fourier series' are 'infinite sums'. Second, not every Hilbert space is separable, i.e. this means that orthonormal bases need be countable, if we do not want to restrict ourselves to separable Hilbert spaces, we will encounter infinite sums with non-denumerable index sets. In the next definition we explain what we mean by

Definition 3.5 *Let H be a normed space with norm $\|\|$. Let $x_1, x_2, ...$ be countable sequence of elements of H. Then we say the series*

Proposition 3.1 *Let $\{e_n, n \in N$ be a countable orthonormal system and $x \in H$. Then we have*

$\sum |< x, e_n >|^2 \leq \| x \|^2$

Note that the series above converges absolutely. Rearranging terms neither affects convergence nor the limit.

The above inequality is known as *Bessel's inequality.* Bessel's inequality has the following interesting consequence.

Corollary 3.1 *Given any orthonormal system S and $x \in H$. Then $S_x =: \{e \in S \,|< x, e >\neq 0\}$ is at most countable.*

This can be seen as follows. Consider the sets $T_{x,n} =: \{e \in S \,||< x, e >|\geq 1/n\}$. By Bessel's inequality these sets are finite and thus $T = \bigcup_{n\in N} S_{x,n}$ is finite or countable.

Let us now recall the concept of convergence in the norm. Given a normed space H with norm $\|\|$ ad a sequence of vectors $x_n, n = 1, 2....$ We say the series ${}_n x_n$ converges in the norm to x, in symols $x =_n^\infty x_n$ if for any $\epsilon < 0$ there exists an n_0 such that $\| x - \sum_{n=1}^\infty x_n \|< \epsilon$ for all $n \geq n_0$.

The significance of [**?**] is that for any orthornomal system S we can make sense of a series of the form $\sum_{e\in S} < x, e > e$. It's always a 'countable' sum.

Theorem 3.3 *Let S be an orthonormal basis and suppose that $\sum \mid \alpha_e \mid^2 < \infty$ with $\alpha_e =< x, e >$ for all $e \in S$. Note again that this series coverges absolutely. Then*
$\sum_{e \in S} \alpha_e e$ *converges absolutely (in the norm) with limit x.*

Theorem 3.4 *Let $S \subset H$ be an orthonormal system. Then there exists an orthonormal basis T such that $S \subset T$.*
The following conditions are equivalent:

- *(i) S is an orthonormal basis.*
- *(ii) If $x \in H$ is orthogonal to S then $x = 0$.*
- *(iii) H is the closure of the subspace spanned by S, $S=\overline{linS}$*
- *(iv) Any $x \in H$ has a Fourier expansion in therms of S. This means that $x = \sum_{e \in S} < x, e > e$.*
- *(iv) For any $x, y \in H$, $< x, y >= \sum_{x \in S} < x, e >< e, y >$*

As to the 'infinite sums' in the above theorem note that these series are actually 'countable' sums since by [**?**] only countably many members are non-zero. Moreover these series are invariant under any permutation of the index set in the sense that 'rearranging' terms neither affects convergence nor the limit of the series. Call such series absolutely convergent.

Theorem 3.5 *Let S be an orthonormal system. Then for any x the series $\sum_{e \in S} < x, e > e$ converges absolutely to the orthogonal projection of x on $\overline{linS}$.*

As a consequence we have the following useful observation. Given an orthonormal system $x_{i i \in I}$ and let A be the smallest closed subspace containing it, then the elements of A are precisely those having the form $\sum_{i \in I} \alpha_i x_i$ with $\sum_{i \in I} \mid \alpha_i \mid\}^2 < \infty$.

It can be proved that if two closed subspaces A and B of a Hilbert space are orthogonal, then $A + B$ is closed. Generally, however, this is not true.

We now give an example for two closed subspaces A and B of an infinite dimensional Hilbert space such that $A + B \neq A \vee B$. For this first note that in an infinite dimensional Hilbert space we may find two orthonormal sequences $x_{n n \in N}$ and $y_{m m \in N}$ such that $x_n \perp y_m$. In the construction below we closely follow Halmos' book [26]. Consider the sequence $\mathrm{z}_n =: \cos(1/n) x_n + \sin(1/n) y_n$. By the Pythagorean theorem we have $\parallel z_n \parallel^2 = \cos^2(1/n) + \sin^2(1/n)01$. Moreover, a straightforward calculation shows $< z_n, z_m > = 0 for n \neq m$. Thus the sequence z_n is an orthonormal system. Let A and B be the smallest closed subspace containing x_n and z_n respectively. Since $\cos(1/n) \neq 0$ we have $y_n \in A + B$. Now note that $\sum_{n=1}^{\infty} \sin^2(1/n) < \infty$. By ? $y = \sum_{i=1}^{\infty}$ makes sense and is an element of $A \vee B$. We now prove that y is not in $A + B$. Suppose $y \in A + B$. Then we would have $y = x + z$ with $x \in A$ and $z \in B$. Using the fact that $< z_n, y_m >= \delta_{nm}$ we get $\sin/1/m) =<$

$y, y_m >=< x + z, y_m >=< z, y_m >=< \sum_{i=1}^{\infty} < z, z_n >< z_n, y_m >=< z, z_m ><$ $z_m, y_m >=< z, z_m > \sin(1/m)$. It would follow that for all $m < z, z_m >= 1$, which contradicts the fact that the sequence of Fourier coefficients $< z, z_m >$ $tendstozero. We have therefore found a y such that y \in A \vee B$ but not $y \in A + B$.

The above example may also serve as an example of a subspace of an infinite dimensional Hilbert space which is not closed.

3.4 More Lattice Theory

Definition 3.6 *The lattice $\mathcal{L}$ is called modular if the modular condition (MC) holds:*

$$\textit{If } A \leq B \textit{ then } A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

If $A \leq B$ then $A \vee B = B$ and thus the modular condition is equivalent to the following:

$$\text{If } A \leq B \text{ then } A \vee (B \wedge C) = B \wedge (A \vee C)$$

Definition 3.7 *An orthocomplemented lattice is called an orthomodular lattice if the orthomodular condition (OMC) holds: If $A \leq B$ and $A^{\perp} \leq C$ then $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ In fact, in this case we have $A \vee (B \wedge C) = B$*

There are various equivalent definitions of orthomodularity, see for instance Redei's book [54]

The following version will be of use later in the book.

Proposition 3.2 *An orthocomplemented lattice is orthomodular iff $A \leq B$ implies $B = A \vee (A^{\perp} \wedge B)$.*

Note that distributivity implies modularity and modularity implies orthomodularity. The converse implications do not hold.

We write $A < B$ for $A \leq B$ and $A \neq B$.

Definition 3.8 *Given any lattice $\mathcal{L}$ and $A, B \in \mathcal{L}$. We say that B covers A if $A < B$ and $A < C < B$ is satisfied by no C.*

An element $A \in \mathcal{L}$ is called an atom if it covers 0. $\mathcal{L}$ is called atomic if for any $B \in \mathcal{L}$ there exists an atom A such that AB. $\mathcal{L}$ is called atomistic if any element B is equal to the least upper bound of those atoms A satisfying $A \leq B$. We say that $\mathcal{L}$ has the covering property if the following holds. Let P be an atom and $A \in \mathcal{L}$. Then $A \vee P$ covers A. We say that A commutes with B if $A = (A \wedge B) \vee (A \wedge B^{\perp})$. Call the set of all elements of $\mathcal{L}$ commuting with all others the center of $\mathcal{L}$. If the center of $\mathcal{L}$ consists of 0 and 1 only we call $\mathcal{L}$ irreducible.

Remark: In the lattice of projections of a Hilbert space two projections commute in the sense above iff they commute as operators.

Definition 3.9 *Given any poset $\mathcal{P}$ we may define the notion of a chain and that of the length of a chain in a straightforward way. We then define the height of $\mathcal{P}$ denoted by $h(P)$ as the supremum over the lengths of all chains of $\mathcal{P}$ minus 1.*

Proposition 3.3 *In an orthocomplemented lattice we have the de Morgan rules:*

- $(\bigvee_i A_{i))}^{\perp} = \bigwedge A_i{}^{bot}$
- $(\bigwedge_i A_{i))}^{\perp} = \bigvee A_i{}^{\perp}$

In the sequel we will use the term 'polynomial'. For this note that $\wedge$ and $\vee$, $\perp$ may be viewed as algebraic operations. The term 'polynomial' may then be defined as usual in algebra. By a (lattice) conditional we mean a polynomial in two variables. For instance, given two elements A and B of an orthocomplemented lattice, then the polynomial $S(A,B) =: A^{\perp} \vee B$ represents a conditional, namely 'material implication'.

Lemma 3.1 *Let L be an orthocomplemented lattice. Suppose there exists a conditional $S(A,B)$ satisfying the following condition.*

$$A \wedge C \leq B \textit{ iff } C \leq S(A,B$$

Then $\mathcal{L}$ is a Boolean algebra.

Proof.

Fr the sake of convenience we write $A \rightsquigarrow B$ for $S(A,B)$. Given any elements A, B, C. We need to show that $((A \vee B) \wedge C) = (A \wedge C) \vee (B \wedge C)$. For this it suffices to show that $((A \wedge B) \vee C) \leq (A \wedge C) \vee (B \wedge C)$. We have

$$A \wedge C) \leq (A \wedge C) \vee (B \wedge C)$$

and

$$(B \wedge C) \leq (A \wedge C) \vee (B \wedge C)$$

Using the condition on $S(A,B)$ we obtain

$$A \leq (C \rightsquigarrow ((A \wedge C) \vee (B \wedge C)))$$

and

$$B \leq (C \rightsquigarrow ((A \wedge C) \vee (B \wedge C)))$$

Hence

$$(A \vee B) \leq ((C \rightsquigarrow ((A \wedge C) \vee (B \wedge C)))$$

Applying the condition on $S(A,B)$ in the other direction we get

$$(A \vee B) \wedge C \leq (A \vee B) \wedge (A \vee B)$$

This is what we wanted to prove.

■

Theorem 3.6 (Mittelstaedt) *Let $\mathcal{L} = \langle L, \leq, ^{\perp} \rangle$ be an orthocomplemented lattice. Then $\mathcal{L}$ is orthomodular if there exists a conditional $S(A, B)$ such that the following conditions are satisfied.*

$$\textit{(i)}\ A \wedge S(A, B) \leq B$$

$$\textit{(ii)}\ A \wedge C \leq B \textit{ implies } A^{\perp} \vee (A \wedge C) \leq S(A, B)$$

A conditional satisfying the above conditions is unique, namely

$$S(A, B) = A^{\perp} \vee (A \wedge B).$$

$\mathcal{L}$ is a Boolean algebra if the above conditions are satisfied by 'material implication', i.e. $S(A, B) = A^{\perp} \vee B$.

We denote $A^{\perp} \vee (A \wedge B)$ by $A \rightsquigarrow_s B$.

Proof. Assume there exists a conditional satisfying (i) and (ii). We need to show that $\mathcal{L}$ is orthomodular. So let $B \leq A$ and $C \leq A^{\perp}$. Then we have

$$B = B \wedge A \leq A^{\perp} \vee (A \wedge B) = A \rightsquigarrow_s B)$$

Moreover we have

$$C \leq A^{\perp} \leq A^{\perp} \vee (A \wedge B) = A \rightsquigarrow_s B$$

Hence

$$B \vee C \leq A \rightsquigarrow_s B$$

and thus

$$A \wedge (B \vee C) \leq A \wedge (A \rightsquigarrow_s B)$$

By condition (i) we have

$$A \wedge (A \rightsquigarrow_s B) \leq B$$

It follows that

$$A \wedge (B \vee C) \leq B$$

and by the definition of orthomodularity this means that $\mathcal{L}$ is orthomodular.

For the proof of uniqueness let $S'(A,B)$ be any conditional satisfying (i) and (ii). We can then assume orthomodularity of $\mathcal{L}$. Putting $C = B$ we get by (ii) that $A \rightsquigarrow_s B \leq S'(A,B)$. Since $S'(A,B)$ satisfies (i) it follows that $A \wedge S'(A,B) \leq (A \wedge B) \leq A^\perp \vee (A \wedge B) = A \rightsquigarrow_s B$. Hence $A^\perp \vee (A \wedge S'(A,B)) \leq A \rightsquigarrow_s B)$. Considering that $A^\perp \leq S'(A,B)$ we have by orthomodularity that $A^\perp \vee (A \wedge S'(A,B)) = S'(A,B)$. It follows that $S'(A,B) \leq A \rightsquigarrow_s B$. $S'(A,B)$ and $A \rightsquigarrow_s B$ are thus equal.

We still need to prove that if conditions (i) and (ii) are satisfied by $S(A,B) =: A^\perp \vee B$, $\mathcal{L}$ is a Boolean algebra. For this we have to verify the condition in the preceding lemma. First note that in this case $A^\perp \vee (A \wedge B) = A^\perp \vee B$. So assume $A \wedge B \leq C$. Then it follows from condition (ii) of this theorem that $A^\perp \vee C \leq A^\perp \vee B$. Hence $C \leq A^\perp \vee B$. For the other direction assume $C \leq A^\perp \vee B$. Then we have $(A \wedge C) \leq A \wedge (A^\perp \vee B)$. By condition (i) we have $A \wedge (A^\perp \vee B) \leq B$. Thus $(A \wedge C) \leq B$. This completes the proof.

■

3.5 The lattice of closed subspaces and projections of an orthomodular space

In this section we do not restrict ourselves to Hilbert spaces. Rather we always have in mind the more general case of an *orthomodular space*. The concept of an orthomodular space is more general than that of a Hilbert space, but it suffices for many purposes. Unless explicitly mentioned otherwise the spaces under consideration in this section are orthomodular spaces.

Definition 3.10 *Let K be a (not necessarily commutative) field with an involution τ, i.e. a function $\tau : K \to K$ such that*

$$\tau(a+b) = \tau(a) + \tau(b), \tau(ab) = \tau(b)\tau(a), \ \tau\tau(a) = a$$

Now, let H be a vector space over K and $\langle\rangle : H \times H \to K$ be a Hermitian form on H, i.e. $\langle\rangle$ satisfies

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$$

$$\langle z, ax + by\rangle = \langle z, x\rangle\tau(a) + \langle z, y\rangle\tau(b)$$

$$\langle x, z\rangle = \tau(\langle z, x\rangle)$$

then call the pair $\langle H, \langle\rangle\rangle$ a Hermitian space. *Call $\langle\rangle$* anisotropic *iff*

$$\langle x, x\rangle = 0 \textit{ implies } x = 0$$

Define the concepts of orthogonality *of vectors and the orthogonal complement $U^\perp$ of a subspace U as in the case of Hilbert spaces. Call a subspace U* closed *iff $U = U^\perp$.*

Definition 3.11 *Call a Hermitian space* $\langle H, \langle\rangle\rangle$ *an* orthomodular space *iff for every closed subspace* U *we have*

$$H = U \oplus U^{\perp}$$

Every Hilbert space H is an orthomodular space. A subspace of H is closed in the sense of the topology of Hilbert space iff it is closed in the sense of an orthomodular space as defined above.

We denote the set of closed subspaces of H as $Sub(H)$. It is obvious that set inclusion is a partial order on $Sub(H)$. Given $M, N \in Sub(H)$, then $M \cap N$ is the greatest lower bound of M and N. The lowest upper bound of N and M denoted by $M \vee N$ is given by the smallest closed subspace containing M and N. Orthogonal complement formation is an orthocomplementation of $Sub(H)$. The zero space is the null element of the lattice and H is the unit element. $Sub(H)$ is thus an orthocomplemented lattice.

We can, as in the case of a Hilbert space, associate a projection with every closed subspace. The set of projections forms, in a natural way, an orthocomplemented lattice (ortho)- isomorphic to the lattice of closed subspaces. An orthoisomorphism between orthocomplemented lattices is a bijctive mapping respection all 'latice operations' including orthocomplementation. In the sequel we mean ortho-isomorhism (automorphism) whenever we use the term isomorphism (automorphism).

The following proposition generalises an observation made by Hardegree in [44]. in connection with Hilbert spaces.

Proposition 3.4 *Let* H *be an orthomodular space,* $x \in H$*,* $A, B \in Sub(H)$*. Then* $Ax \in B$ *iff* $x \in A^{\perp} \vee (A \wedge B)$

Proof. First note that the closed subspaces $A^{\perp}$ and $A \wedge B$ are orthogonal. Then we have $A^{\perp} \vee (A \wedge B = A^{\perp} \oplus (A \wedge B)$.

For the direction from left to right let $x = y + z$ be the unique decomposition of x with respect to A and $A^{\perp}$, i.e. $y \in A$ and $z \in A^{\perp}$. We have $Ax = y$. The hypothesis says that $y \in B$. Thus $y \in A \wedge B$. It follows that $x \in A^{\perp} \oplus (A \wedge B)$.

For the direction from right to left observe $A^{\perp} \oplus (A \wedge B)$ is again an orthomodular space with the Hermitian form properly restricted. We have thus, in addition to the above decomposition, a decomposition $x = y_1 + z_1$ with $y_1 \in A$ and $z_1 \in A \wedge B$. Since the decomposition is unique we have $y = y_1$ and $z = z_1$. It follows that $Ax = y = y_1 \in B$. ■

We call, for historical reasons, a lattice a Hilbert lattice if it is isomomorphic to the lattice of closed subpaces of an orthomodular space.

Theorem 3.7 *A Hilbert lattice is an atomistic, complete, orthomodular irreducible lattice having the covering property.*

The following theorem is *Piron's Representation Theorem*

Theorem 3.8 (Piron) *An ortholattice $\mathcal{L}$ of height ≥ 4 is a Hilbert lattice iff it is atomistic, complete, irreducible, orthomodular and it has the covering property.*

Theorem 3.9 (Amemiah-Araki) *A Hilbert H is finite dimensional iff $Sub(H)$ is modular.*

It is an important fact that for any infinite-dimensional Hilbert space H, $Sub(H)$ is orthomodular but *not* modular.

3.6 Characterising classical Hilbert lattices

In quantum mechanics we are (primarily) concerned with infinite-dimensional Hilbert spaces. We define, for historical reasons, a *classical Hilbert lattice* to be a lattice isomorphic to the lattice of closed subspaces of an infinite - dimensional Hilbert space. Recall that we have already defined a *Hilbert lattice* to be a lattice isomorphic to the lattice of closed subspaces of some orthomodular space. For these lattices we have Piron's representation theorem which characterises Hilbert lattices of height at least 4.

In this section we characterise classical Hilbert lattices among ortholattices. For this purpose we need, apart from Piron's theorem, three deep theorems of modern Hilbert space theory, namely the theorems of Solèr, Wigner and Mayet, which we state below.

In his pionering paper [36] Keller settled a long standing question, namely the question whether every infinite dimensional orthomodular space was already a Hilbert space. Keller's ingenious construction of a counter example settled the question in the negative. This, however, posed another problem, namely the problem of characterising those orthomodular spaces that are in fact Hilbert spaces. This problem was solved by Maria Pia Solèr in [61].

Theorem 3.10 (Solèr) *Let $\langle H, \langle\rangle\rangle$ be an orthomodular space over K and let $c \in K$. Suppose there exists an infinite family $(x_i)_{i\in I}$ of pairwise orthogonal elements of H such that for all $i \in I$, $\langle x_i, x_i\rangle = c$. Then K must be the (skew-) field of the real numbers,the complex numbers or quaternions and H is an infinite - dimensional Hilbert space.*

Definition 3.12 *Let H_1 and H_2 be two orthomodular spaces and $\sigma : H_1 \to H_2$ be a bijective map. We say that σ is a semiunitary map iff the following conditons are satisfied.*

- *For any $x, y \in H_1$, $\sigma(x + y) = \sigma(x) + \sigma(y)$.*
- *There exists an automorphism ρ of K such that, for any $\lambda \in K$ and any $x \in H_1$, we have $\sigma(\lambda x) = \rho(\lambda)\ (\sigma x)$.*
- *There exists $\lambda_\sigma \in K$ such that, for any $x, y \in H_1$, we have $\langle\sigma(x), \sigma(y)\rangle = \rho(\langle x, y\rangle)\lambda_\sigma$.*

If, moreover, we have $\rho = id_K$ and $\lambda_\sigma = 1$, we say that σ is unitary.

Theorem 3.11 (Wigner) *Let H_1 and H_2 be orthomodular spaces of dimension at least 3. Then every ortholattice isomorphism $f : Sub(H1) \to Sub(H_2)$ is induced by some semiunitary map.*

We need the following result by Mayet which, essentially, is a consequence of Wigner's theorem.

Theorem 3.12 (Mayet) *Let H be an orthomodular space of dimension at least 3 and let $X \in Sub(H)$ of dimension at least 2. Let f be an automorphism of $Sub(H)$ whose restriction to $[0, X]$ is the identical map. Then there exists a unique unitary operator σ on H inducing f such that the restriction of σ to X is the identical map.*

Solèr's theorem characterises Hilbert spaces among orthomodular spaces. We are interested in a charactersation of classical Hilbert lattices Hilbert lattices among ortholattices.

The characterisation we give is in terms of a symmetry property.

For a given ortholattice L we call two atoms σ_1 and σ_2 orthogonal if $\sigma_1 \leq \sigma_2^\perp$. This relation is readily seen to be symmetric.

Definition 3.13 *Let L be a complete ortholattice and let $\Delta = (\sigma_i)_{i\in I}$ be an infinite pairwise orthogonal family of atoms of L. We say that L satisfies the symmetry property (synonymously: is symmetric) with respect to Δ iff the following holds. For any permutation $f : I \to I$ there exists an ortholattice automorphism ρ_f of L with the following properties.*

- *ρ_f extends f, i.e. $\rho_f(\sigma_i) = \sigma_{f(i)})$ for any $i \in I$.*
- *If the set J of those elements of I which are left fixed by f is non-empty, ρ_f induces the identical map on $[0, A]$, where A denotes the least upper bound of the family $(\sigma_j)_{j\in J}$.*

We say that L is symmetric iff there exists an infinite pairwise orthogonal family Δ of atoms of L such that L is symmetric with respect to Δ.

Theorem 3.13 *A Hilbert lattice L is a classical Hilbert lattice iff it is symmetric.*

Proof: Let us first verify that for a given infinite - dimensional classical Hilbert space H $Sub(H)$ is symmetric. To see this consider a complete orthonormal system $(x_i)_{i\in I}$ of H. Then the family of one - dimensional subspaces $(\langle x_i\rangle)_{i\in I}$ is an infinite orthogonal system of $Sub(H)$. Let $f : I \to I$ be any permutation of I. Recall that $x = \sum_{i\in I}\langle x, x_i\rangle x_i$. Define the map φ_f as follows. For $x = \sum_{i\in I}\langle x, x_i\rangle x_i$ put $\varphi_f(x) =: \sum_{i\in I}\langle x, x_{f^{-1}(i)}\rangle x_i$. φ is well defined. For any $i \in I$ we have $\varphi_f(x_i) = x_{f(i)}$. Moreover, φ_f is unitary, since for any $x, y \in H$ we have $\langle\varphi_f(x), \varphi_f(y)\rangle = \sum_{i\in I}\langle x, x_{f^{-1}(i)}\rangle\overline{\langle y, x_{f^{-1}(i)}\rangle} = \sum_{i\in I}\langle x, x_i\rangle\overline{\langle y, x_i\rangle} = \langle x, y\rangle$.

Suppose $\{i \mid f(i) = i\}$ is non-empty and denote by X the smallest closed subspace containing $\{x_i \mid f(i) = i\}$. X is the smallest closed subspace containing $\{\langle x_i \rangle \mid f(i) = i\}$ and φ_f induces the identity on X. For the latter claim observe that φ_f induces the identity on the subspace spanned by $\{x_i \mid f(i) = i\}$ and and X is the closure of that subspace. Since φ_f is continuous, it induces the identity on X too. φ_f thus induces an ortholattice automorphism ρ_f on $Sub(H)$ such that for any $i \in I$, $\rho_f(\langle x_i \rangle) = \langle x_{f(i)} \rangle$. Clearly, ρ_f induces the identical map on $[0, X]$. Thus symmetry of $Sub(H)$ is proved.
For the other direction note that the symmetry property implies infinite height. It thus suffices by Piron's Representation Theorem that any orthomodular space H such that $Sub(H)$ has the symmetry property is an infinite - dimensional classical Hilbert space. So let $(\langle x_i \rangle)_{i \in I}$ be an infinite orthogonal family with respect to which $Sub(H)$ is symmetric. Let $i_0 \in I$. For any $j \in I, i_0 \neq j$ consider the permutation f_j of I defined as follows.

$$f_j(i_0) = j,\ f_j(j) = i_0,\ \mathrm{f}_j(i) = i \text{ else.}$$

Denote by X the smallest closed closed subspace of X containing $\langle x_i \rangle$ for all $i \in I$. X is infinite - dimensional. By symmetry there exists an automorphism ρ_j of $Sub(H)$ inducing the identity on $[0, X]$ such that for all $i \in I$, $\rho_j(\langle x_i \rangle) = \langle x_{f_j(i)} \rangle$. So, by Mayet's theorem, ρ_j is induced by some unitary map φ_j. Put $y_j =: \varphi_j(x_{i_0})$ for $j \neq i_0$ and $y_{i_0} = x_{i_0}$. Then, since φ_j is unitary, the family $(y_j)_{j \in I}$ is a family as required in Solèr's theorem. It follows by Solér's theorem that H must be an infinite - dimensional classical Hilbert space. □
As a corollary we get the following theorem, which gives another characterisation of Hilbert spaces among orthomodular spaces.

Theorem 3.14 *Let $\langle H, \langle\rangle \rangle$ be an orthomodular space over K. Then the following conditions are equivalent.*

- *There exists an infinite family $(x_i)_{i \in I}$ of pairwise orthogonal elements of H and a non zero $c \in H$ such that for all $i \in I$ we have $\langle x_i, x_i \rangle = c$.*
- *$Sub(H)$ is symmetric.*
- *H is an infinite - dimensional classical Hilbert space.*

Chapter 4

Basics of the Formalism of Quantum Mechanics

4.1 Some History

It is a common experience among students of physics that their first course on quantum mechanics comes as a sort of shock. In such a course the student is confronted with a discipline that does not display the pattern he is used to from the physical theories he has already mastered such as Newtonian mechanics, electrodynamics, special relativity... The student must swallow the fact that in contrast to these classical physical theories quantum mechanics is essentially a mathematical formalism. He is taught how to make use of this formalism in order to calculate certain physical quantities such as the energy levels of the electron in the hydrogen atom. The success of the formalism of quantum mechanics is unique in the history of science yielding the correct results with unprecedented precision for a vast range of phenomena which were entirely untractable in classical physics. This is the reason for the wide spread slogan that quantum mechanics is the "most successful physical theory ever".

However, the student's question "Why this formalism? Where does it come from?" gets normally, if at all, an evasive and unsatisfactory answer. The plain truth is that this formalism is the result of guesswork, ingenious guesswork admittedly.

The first version of the formalism of quantum mechanics became known as *matrix mechanics*. The first and already crucial step in this process of guessing was taken in June 1925 by Werner Heisenberg, a then 23 year old post-doc, in his famous paper [28]. Essentially, the discovery was that physical quantities such as energy, momentum...are to be represented by infinite matrices in such way that the possible values a physical quantity can assume are given by the eigenvalues of the corresponding matrix. The matrices representing physical quantities were not required to commute and non-commutation of matrices was to be regarded as "non-simultaneous measurability" of the corresponding physical quantities.

It is interesting to note that Heisenberg did not explicitly use the concept of a matrix. In fact, he did not even know that concept at the time. The concept of a matrix was then not part of a mathematician's standard background let alone a physicist's. The technical tools used in the paper were somehow vague as a sort of 'schemata of numbers' that are multiplied in a way reminiscent of the multiplication of Fourier series. It was Heisenberg's teacher Max Born who, after Heisenberg's paper had appeared, realised that these mathematical schemata were actually (infinite) matrices and Heisenberg's way of combining them was actually matrix multiplication. In the joint paper [7] by Born and Jordan, which appeared only a few months after Heisenberg's paper, the first precise account of matrix mechanics was given for systems with one degree of freedom. In particular it contains the first mathematically precise statement of Heisenberg's Uncertainty Principle: If P is the matrix representing the position of a particle and Q is the matrix representing its momentum we have

$$PQ - QP = h/\pi i$$

In the subsequent classic paper by Born-Heisenberg-Jordan [6] the formalism of matrix mechanics was fully established. The concept of a Hilbert space, however, still had no place in this. This had to wait for John von Neumann's work in the late twenties, see [62], [63] and in particular his classic book [64] of 1932 in which the Hilbert space formalism of quantum mechnics received its final elegant shape.

It is interesting that similar guesswork led to Schrödinger's wavemechanics which later on was proved to be equivalent to matrix mechanics.

In 1926 Erwin Schrödinger published his famous paper [58], which, essentially earned him the Nobel Prize as did Heisenberg's paper [28] for Heisenberg. In this paper he claimed and proved to have found a (partial) differential equation with a remarkable property. This differential equation has become known as the Schrödinger equation, which nowadays is probably the mnost famous equation of physics. And what was so remarkable about it? Well, the claim Schrödinger made and proved was that his equation had square integrable solutions exactly for those eigenvalues that correspond to the experimentally found energy levels which the electron in the hydrogen atom can assume. This is all Schrödinger claimed and proved. The solution of Schrödinger's equation usually denoted by $\Psi(x, y, z)$ is (in the stationary case) a complex valued functions of the three spatial coordinates x, y, z and in the non-stationary case also of time t. Schrödinger had, at the time when his paper appeared, no idea of the physical meaning of the function Ψ. The nowadays generally accepted physical interpretation given to the Ψ function by Born in [5] is that $\mid \psi \mid^2 dxdydz$ represents the probability with which the particle can be found in the infinitesimal volume $dxdydz$. So, according to this interpretation, the solutions of the Schrödinger equation are *probability waves*. It is an irony of the history of science that Schrödinger himself never accepted this interpretatiom of the function Ψ.

To summarise, the core of quantum mechanics is a formalism, and this formalism was *guessed*. It always works with amazing precision. One of the aims

we pursue in this book is to put logic to good use in order to shed light on this formalism.

4.2 Hermitian operators

We will see that certain operators on Hilbert spaces play a vital role in the formalism of quantum mechanics. These operators are called *Hermitian operators.*

Given a Hilbert space H. Then we call any linear map of H into itself an operator of H. Let T be an operator of H. We call T bounded if there exists a positive real number c such that for all $x \in H$ we have $\parallel Tx \parallel \leq c \parallel x \parallel$.

One can then prove that an operator is bounded iff it is continuous. It can also be proved that given any bounded operator T there exists a unique operator T^* such hat for all x and y we have $\langle Tx, y\rangle = \langle x, T^* y\rangle$. T^* is called the adjoint of T.

Definition 4.1 *Call a bounded operator T Hermitian if $T = T^*$. Call a Hermitian operator T positive definite if for any $x \in H$ we have $\langle x, Tx\rangle \geq 0$. Call a bounded operator T unitary if T is bijective and for any x and y we have $\langle Tx, Ty\rangle = \langle < x, y\rangle$*

It can then be proved that a unitary operator may also be defined as a bijective Hermitian operator the adjoint of which is its inverse.

Proposition 4.1 *Let T be an Hermitian operator. Then*

- *There exists an orthonormal basis of eigenvectors of T.*
- *The eigenvalues of T are real numbers.*
- *All eigenvalues are positive iff T is positive definite.*

From this it follows that, given a Hermitian operator T, any vector has a Fourier expansion in terms of eigenvectors of T.

Proposition 4.2 *Let S and T be Hermitian operators. Then the following conditions are equivalent*

- *S and T commute, i.e. $ST = TS$*
- *ST is a Hermitian operator.*
- *There exists an orthonormal basis which is a family of eigenvectorswhich is common to boh S and T.*

4.3 Postulates of Quantum Mechanics

In this section we make the connection between Hilbert space and quantum mechanics. We describe the basics of the formalism of quantum mechanics. In this we heavily rely on the excellent textbook [9].

Let us first keep in mind the following fact which marks a main difference beween classical and quantum physics. Given a physical system $\mathcal{S}$ in a certain state and let $\mathcal{A}$ be any physical quantity pertaining to $\mathcal{S}$. Suppose a measurement of $\mathcal{A}$ is performed. On the view of classical mechanics $\mathcal{A}$ *possesses* a certain 'true' value in every state of S and the function of measurement consists in 'finding out' this value. There is no reason to assume that the process of a measurement should change the state of the system. In fact, this view is completely alien to classical mechanics. This view cannot be maintained in quantum mechanics. Rather, according to quantum mechanics, we may in general get a variety of values as a result of measurement each with a certain probabilty which depends on the state of the system. We will state the precise rule for this below as part of the mathematical formalism. In general the state of the system undergoes change in the process of measurement. Every subsequent measurement of $\mathcal{A}$, however, leaves the state unchanged and yields the same value as the initial measurement.

Gven any physical system $\mathcal{S}$. Then, according to quantum mechanics, we can associate with $\mathcal{S}$ a Hilbert space H such in such a way that there is a one-to-one correspondence between the states of $\mathcal{S}$ and the rays. i.e. the one-diensional subspaces of H. The physical quantities of $\mathcal{S}$ such as energy, angular momentum...are represented by Hermitian operators of H. If a physical quantity of $\mathcal{S}$ is represented by the Hermitian operator T then the values this quantity can assume are precisely the eigenvalues of T, which, as eigenvalues of a Hermitian operator, are real numbers. Suppose this observable is measured. Then after measurement the system is in a state which is corresponding to an eigenvector of T, more precisely by the ray spanned by this eigenvector.

The mathematical formalism of quantum mechanics provides the answers to the follwing questions..

1) How is the state of a physical system $\mathcal{S}$ represented mathematically?

2) What are the possible outcomes of measurements in a given state?

3) How does a given state change in the process of measurement?

3)How does the system evolve over time?

Let us now present some of the postulates of the formalism of quantum mechanics.

This still needs some extension which can be done routinely

Postulate 1 *At any fixed time there exists a one-to-one correspondence between the states of $\mathcal{S}$ and the rays of H.*

Postulate 2 *Every physical quantity* $\mathcal{A}$ *of* $\mathcal{S} corresponds to an Hermitian operator A of H.$
We call the Hermitian operator A the observable corresponding to $\mathcal{A}$.

Postulate 3 *The possible results of the measurement of a physical quantity* $\mathcal{A}$ *are eigenvalues of the corresponding observable* A.

Note that Hermitian operators may have a continuous set of eigenvalues rather than a discrete one. For the of sake of simplicity we formulate the following postulates only in the discrete case. In the discrete case we again have to distinguish between two cases, the non-degenerate case and the degenerate case. We have the degenerate case whenever the eigenspace corresponding to some eigenvalue has dimension greater than one otherwise we have the non degenerate case.

Postulate 4 (non-degenerate case) *When the physical quantity* $\mathcal{A}$ *is measured in the normalised state* x*, the probability of obtaining the (non-degenerate) eigenvalue* a_n *is given by*

$$P(a_n = |< y_n, x >|^2,$$

where y_n *is the normalised eigenvector corresponding to the eigenvalue* a_n.

The following postulate is called the *Projection Postulate*

Postulate 5 *Suppose the system is in state* $\langle x \rangle$ *and a measurement of the quantity* $\mathcal{A}$ *is performed in this state. Suppose we get as a result of this measurement the value* a*. By Postulate 2* a *is an eigenvalue of* A*. Denote by* P_a *the projection onto the eigenspace corresponding to the eigenvalue* a*. Then the state of the system 'after measurement' is* $\langle y \rangle$ *where* $y = P_a(x)$.

The reader will miss an important postulate here, namely the postulate concerning the combination of two systems. this postulate says hat the combination of two systems is represented by the tensor product of their respective Hilbert spaces. The reason is that we still have to think about the way we will introduce the tensor product of Hilbert spaces. On the one hand we do not have the highly non-elementary resources available in this book to this in a way meeting the highest mathematical standards. On the other hand we do not want to do this in the very loose form adopted in most textbooks on quantum mechanics. We try, for didactic reasons, to find a way of introducing the tensor product in a way which is somewhere in the middle between these extremes. We will need the tensor product again in Chapter 11.

Chapter 5

Birkhoff- von Neumann 1936

Abstract:. In this chapter we attempt a detailed analysis of the classic 1936 paper by Birkhoff-von Neumann paper(BvN), which initiated quantum logic. We deal extensively with BvN for two reasons. This seminal paper is essential for the understanding of any approach to quantum logic. Second, this paper is, though quoted frequently, not easy to read. It also has an interesting history, as we know from the correspondence between Birkhoff and von Neumann during the writing of the paper. This is of interest in its own right. We will revisit BvN in Chapter..., where we establish the connection with our own work.

We then describe a surprising discovery by Kochen, Specker and Schütte which highligts the peculiar nature of the relationship between quantum logic and classical logic. It says that there exists a classical tautology which is a contradiction in quantum logic. At this stage this phenomenon has no deeper explanation. We will come back to this in Chapter10, where we will generalise this result and look at it in a new light. In a sense, this will turn out to be a special case of the theorem on holistic logics which we call the No Windows Theorem and which will be proved in Chapter 9. ■

It was in 1932 that John von Neumann's classic book "Mathematische Grundlagen der Quantenmechanik" [64] appeared in print. In that book the mathematical formalism of quantum mechanics received its elegant modern form with Hilbert space as its core mathematical structure. In 1936 John von Neumann published a paper entitled "The logic of quantum mechanics" [2] jointly with the Harvard mathematician Garret Birkhoff. This paper marks the birth of what has become known as quantum logic.

Birkhoff and von Neumann's seminal work has something in common with Keynes' famous book "General Theory of Money, Interest and Employment", which initiated Keynesianism. Both works are widely quoted, but not all of those quoting them have actually studied these works in depth. Possibly this is due to the fact that these works were not only highly influential in their

respective fields but that they are also not easily readable, to say the least.

The paper is frequently quoted as introducing the lattice of closed subspaces of a Hilbert space as the core algebraic structure of the 'logic of quantum mechanics'. This is a poor description of this important work. In this section we attempt a detailed analysis of this paper. This is, in view of the importance of this work, an end in itself, but it is also fruitful for making the connection with the approach adopted in this book. In this section we will analyse and in a way reconstruct and interpret this work quoting extensively from the paper itself.

In this enterprise Redei's article [55] "The prenatal history of quantum logic" is extremely useful. It gives insight into the correspondence between Birkhoff and von Neumann during the writing of the paper which cannot be gained from the paper itself. There are some highly interesting passages in the letters which are relevant to the approach adopted in this book. The analysis given by Redei can be supplemented fruitfully by looking at it from the point of view of this book. We will revisit this topic in Chapter 9.

5.0.1 Structure of the paper

The paper starts with an Introduction and ends with an Appendix. Its core is divided into three parts:
(1) Physical Background
(2) Algebraic Analysis
(3) Conclusions

We will give a summary of each of these parts and will try to make transparent the train of thought and the peculiar reasoning, which, as pointed out by the authors themselves, is not free from heuristic features. Our special interest concerns "Physical Background" where the connection between the 'logic of quantum mechanics' and closed subspaces of a Hilbert space is made.

5.0.2 Novel logical notions in quantum mechanics.

The paper starts, in the Introduction, as follows: "One of the aspects of quantum theory which has attracted the most general attention is the novelty of the logical notions it presupposes. It asserts that even a complete mathematical description of a physical system $\mathcal{S}$ does not in general enable one to to predict with certainty the result of an experiment on $\mathcal{S}$, and that in particular one can never predict with certainty both the position and the momentum of $\mathcal{S}$ (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations cannot be made on $\mathcal{S}$ simultaneously (Principle of Non-commutativity of Observations."

This is worth reflecting on. The authors start by saying that quantum theory presupposes new logical notions unfamiliar from classical logic. Then two examples for such "novel logical notions" are given. The first example is Heisenberg's Uncertainty Principle, and the second example is what the authors call the Principle of Non-Commutativity of Observations. At first glance these principles seem to be purely physical in nature. What is interesting here is

that Birkhoff and von Neumann obviously consider these principles not just as (novel) physical principles, which they undoubtedly are, but also as (novel) *logical* notions. This is in fact remarkable.

The paper continues as follows: "The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to set products, linear sums and orthogonal complements -and resembles the usual calculus of propositions with respect to *and*, *or* , and *not*".

Again, this passage is worth reflecting on. What do Birkhoff and von Neumann mean by saying that quantum mechanics does not conform to classical logic? Why does it not conform to classical logic? Is it not true that physicists use classical logic in reasoning about quantum systems? And in fact, Popper for instance does not share the view that quantum mechanics does not 'conform' to classical logic. In [51] he says: "...physical theories, including quantum mechanics, do conform to classical logic, even according to Birkhoff and von Neumann's proposal."

There is a letter written by von Neumann to Birkhoff dated November 2 1935 -the paper was received by Annals of Mathematics on April 4, 1936.- which may cast light on this. John von Neumann writes: "Looking at the paper now I see, that I forget to say this...: That while common logics did apply to quantum mechanics, if the notion of simultaneous measurability is introduced as an auxiliary notion, we wished to construct a logical system, which applies directly to quantum mechanics-without any extraneous secondary notions like simultaneous measurability. And in order to have such a consequent, one-piece system of logics, we must change the classical class calculus of logics."

This passage is crucial to the understanding of the Birkhoff-von Neumann enterprise. From the modern point of view one can reconstruct this as follows. Given a (formal) language $\mathcal{L}_1$ that permits us to make statements about classical mechanics. Then these statements (propositions) are expected to obey classical logic. In such a language there is no need for talking about compatibility or incompatibility of propositions or simultaneous (non-simultaneous measurability) because all propositions are compatible and there is no 'non-simultaneous measurability'. In quantum mechanics, however, this distinction does matter. We might then think of constructing a richer language say by introducing additional operators reflecting non-simultaneous measurability (non-compatibility) of propositions into the language that would allow statements about compatible (incompatible) and simultaneously (non-simultaneously) measurablable observables in quantum mechanics. There is no reason to believe that such statements should not 'conform' to classical logic, at least in the sense that the propsitional connectives combining them should behave classically. And, in fact, the (informal) language physicists use in reasoning about quantum systems is of such a nature, and the (propositional) logic they use is classical logic. Birkhoff and von Neumann were well aware of this, and it is important to note that this is

exactly what they did not want in building 'the logic of quantum mechanics' as is obvious from John von Neumann's letter quoted above.

Phrased in modern terminology, Birkhoff and von Neumann want to retain the *language of propositional logic* as the language of the 'logic of quantum mechanics'. This is what they mean by a calculus that "resembles the usual calculus of propositions with respect to *and*, *or* and *not*". The 'novel logical notions' mentioned are to be reflected in the propositional *(quantum) logic* to be constructed. This logic of course cannot be expected to be classical.

Note that the term 'calculus' is obviously not used in the modern sense. It becomes evident from the further progression of the paper that what the authors have in mind is not (necessarily) a deductive system. Rather it is reminiscent of algebraic logic where logics can be defined using algebraic structures. In algebraic logic, Boolean algebras for instance define classical logic. Generally, the paper does not display the distinction between syntax and semantics common in presenting logical systems in modern style.

In any case it seems that Birkhoff and von Neumann envisage a logical system in (whatever sense) or (in their terminology) logical structures into which notions such as non-commutativity (of observations) or non-simultaneous measurarability, i.e. notions which at first glance seem to have nothing to do with logic, can be incorporated. As indicated these novel features of quantum mechanics should be part of the *logic* and not of the *language of the logic*. It seems that this view of the uncertainty relations as being logical in nature is the main intuition of the Birkhoff-von Neumann paper. This is an insight which plays a vital role in the approach to quantum logic taken in this book.

The quotations above are from the Introduction of the paper.

Let us now take a closer look at the proper contents of the paper.

5.0.3 Experimental Propositions

In the first paragraphs of "Physical Background" the authors set out to explicate the concept of an *experimental proposition*. In this they build on various other concepts. First there is the concept of a *physical* system which is unproblematic and taken for granted by the authors. Another basic concept is that of a *set of compatible measurements*. We may think of this as "single composite measurement", in modern terminology a set of mutually compatible observables represented by mutually commuting Hermitian operators. They then proceed to the concept of an *observation* on $\mathcal{S}$: Let $\mu_1, ...\mu_n$ be n compatible measurements with outcomes $x_1, ..., x_n$. The observation amounts to specifying these values. Call the set of all n-tuples that can arise as values in compatible measurements an *observation space* of the system. Note that the concept of an observation space is relative to a finite number of compatible observations. So, in case $n = 1$, we have just one observable and the observation space coresponding to this observable is the set of all possible values it can assume as a result of an experiment. It is then natural to define an experimental proposition to be a subset of an observation space. We may view such an experimental proposition as a sort of prediction saying that the value of an observable that we get in a

certain experiment belongs to a certain subset of the observation space. It is made clear, however, in the paper that not every subset of observation space is a proper candidate for this. It would for instance be absurd, as mentioned by Birkhoff and von Neumann, to call the assertion that the angular momentum of of the earth around the sun was at a particular instant a rational mumber an experimental proposition. What, however, is important to note is that in classical physics those subsets of observation space that do represent experimental propositions must form a Boolean algebra with respect to the usual set operations. This reflects the requirement that classical mechanics 'conforms' to classical logic. Those subsets of observation space that actually represent experimental propositions have later been described as Borel measurable sets.

The next crucial concept is that of a *phase space*. In classical mechanics, phase space means the following. Given a physical system of, say n particles. Then the 'state' of this system is characterised by the positions and momenta of these particles. Using the words of Birkhoff and von Neumamm: "Thus, in classical mechanics, each point of Σ corresponds to a choice of .. n position and momentum coordinates... Hence in this case Σ is a region of $2n$-dimensional space."

What, now, is the analogue of this in quantum mechanics according to Birkhoff and von Neumann? They say: "Similarly, in quantum theory the points of Σ correspond to so-called wave functions, and hence Σ is again a function-space, usually assumed to be a Hilbert space."

Hilbert space as the phase space in quantum mechanics enters the stage here by way of analogy. The (heuristic) argument is this. The states of a classical system are determined by a tuple of positions and momenta and the phase space of the system is therefore the set of these tuples. In quantum mechanics the state of the sytem is determined by a wave function and therefore its phase space is the space of its wave functions, which is a Hilbert space.

5.0.4 A propositional calculus for quantum mechanics

From the conceptual point of view, the core of the first part of the paper is the paragraph 6 entitled "A propositional calculus for quantum mechanics", in which the core logical structure of the Birkhoff-von Neumann approach to quantum logic is introduced. It seems to us that this paragraph is not easy to read. We will therefore try to reconstruct the subtle and at times heuristic reasoning in detail so as to get a clear picture of what the authors precisely have in mind. This will help us later on in making the connection between the approach put forward in this book and BvN.

So far there is the concept of an experimental proposition defined as a subset of on observation space. What is needed now is to make the connection between experimental propositions and phase space in quantum mechanics, i.e. Hilbert space. Put differently, the question has to be answered which subsets of a Hilbert space (mathematically) represent experimental propositions. Nowadays, we are familiar with the following answer to this question. An experimental proposition is mathematically represented as a closed subspace of a Hilbert space. Let us

see how Birkhoff and von Neumann arrive at this conclusion.

We quote:" The present section will be devoted to defining such a connection, proving some facts about it, and obtaining from it heuristically by introducing a plausible postulate, a propositional calculus for quantum mechanics".

Note that the authors consider the argument heuristic and that a 'plausible postulate' plays a role in it.

They continue: "Accordingly, let us observe that that if $\alpha_1, ..., \alpha_n$ are any compatible observations on a quantum-mechanical system $\mathcal{S}$ with phase-space Σ, then there exists a set of mutually orthogonal closed linear subspaces Ω_i of Σ (which correspond to the families of proper functions satisfying $\alpha_1 f = \lambda_{i,1} f, ..., \alpha_n f = \lambda_{i,n} f$) such that *every* point (or function) $f \in \Sigma$ can be uniquely written in the form

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + ...[f_i \in \Omega_i]"$$

Let us reconstruct this in modern terminoloy. The above sum is obviously infinite. It is what we nowadays call the Fourier expansion of a vector in terms of a complete orthonormal system, see Chapter 3.3.

Here the term 'compatible observation' must be made precise mathematically. The context suggests that here $\alpha_1, ..., \alpha_n$ represent mutually commuting Hermitian operators. Then what they call the family of 'proper functions' is the family of eigenfunctions (eigenvectors) common to $\alpha_1, ..., \alpha_n$ and $\lambda_{i,1}, ..., \lambda_{i,n}$ are the corresponding eigenvalues. By "..such that..." the fact is expressed that the (normalised) eigenvectors of a Hermitian operator form a complete orthonormal system. In case $n = 1$ the above just says in modern teminology that any vector has a unique Fourier expansion in terms of the eigenvectors of the Hermitian operator representing $\alpha_{1)}$

The text goes on as follows:

"Hence if we state the

Definition 5.1 *By the 'mathematical representative' of a subset S of any observation-space (determined by compatible observations $\alpha_1, ..., \alpha_n$) for a quantum-mechanical system $\mathcal{S}$, will be meant the set of all points f of the phase-space of $\mathcal{S}$, which are linearly determined by proper functions f_k satisfying $\alpha_1 f_k = \lambda_1 f_k, ..., \alpha_n f_k = \lambda_n f_k$, where $(\lambda_1, ..., \lambda_n \in \mathcal{S})$*

Then it follows immediately: (1) that the mathematical repesentative of any experimental proposition is a closed linear subspace of Hilbert space (2) since all operators of quantum mechanics are Hermitian, that the mathematical representative of the *negative* of any experimental proposition is the *orthogonal complement* of the mathematical representation of the proposition itself..." Note that they define the negative of an experimental proposition (or subset S in observation space) to be the experimental proposition corresponding to the set-complement of S in the same observation space.

Again, we think that this needs interpretation. Here, for the first time, the paper uses the term 'closed (linear) subspace'. Why closed subspace? What does 'linearly determined' mean? Assume 'linearly determined' means 'linearly

spanned' in modern terminology. Then the conclusion that the resulting subpaces are closed would not be justified. Recall that in an infinite-dimensional Hilbert space the linear span of a set of vectors need not be closed although we do have this in the finite-dimensional case, see ?.

It seems that the only way of making sense of the above argument is this. Again let us assume the case $n = 1$, i.e. the observation space is determined by one observable. Let α denote the (Hermitian) operator representing this observable and let an experimental proposition $\mathcal{P}$ be given. Thus $\mathcal{P}$ is a subset of the set of eigenvalues of α. Let $\{x_\lambda \mid \lambda \in \mathcal{P}\}$ be the set of eigenvectors corresponding to the elements of $\mathcal{P}$. According to the above definition the mathematical representative of $\mathcal{P}$ the of set of vectors which are 'linearly determined' in Birkhoff and von Neumann's terminology by the $x'_\lambda s$. If, however, we take 'linearly determined' by 'linearly spanned' in modern terminology this span is not necessarily a closed subspace and conclusion (1) would be wrong since the linear span of a set of vectors of a Hilbert space need not be closed in the case of an infinite-dimensional vector space. This is generally true only if this subspace is finite-dimensional. As to terminology birkhff and von Neumann mean by 'Hilbert space' what in modern terminology is an infinite-dimensional Hilbert space. Therefore 'linearly determined' cannot mean 'linearly spanned' in modern terminology. We can, however, make perfect sense of the argument as follows. Take 'linearly determined as meaning to be a finite or infinite sum of the x_λ's, where infinite sum means a Fourier expansion in the x_λ's. Then the set of the "infinite sums" is the boundary of the span of the $x'_\lambda s$ and the resulting space is in fact a closed subspace. On this interpretation conclusion (2) is correct too.

For the sake of clarity let us put it this way. Consider the subspace apanned by the x_λ's. Then by definition any vector of this subspace is a linear combination of the x_λ's. Now take the (topological) closure of this subspace. This is a closed subspace and thus a Hilbert space itself. Any vector of this space has a a Fourier expansion in terms of the x_λ's. It seems to us that this is what Birkhoff and von Neumann mean by "linearly determined". The term means "being a linear combination or the result of a Fourier expansion".

To summarise, the following has been achieved. So far, four meaningful concepts have been introduced in the paper, namely the concepts of an observation space, an experimental proposition, the concept of a phase space, which in quantum mechanics is a Hilbert space, and the concept of the mathematical representation of an experimental proposition. The main conclusion of the partially heuristic but nevertheless convincing reasoning so far is that an experimental proposition should be mathematically represented by a closed subspace of a Hilbert space.

The paper proceeds by introducing the following *Postulate*: "The set-theoretical product of any two mathematical representatives of experimental propositions concerning a quantum-mechanical system, is itself the mathematical representatitive of an experimental proposition"

What does this postulate say? Given any two closed subspaces A and B representing the experimental propositions P and Q. We can then consider

$A \cap B$ which is again a closed subspace. The question is whether there is an experimental proposition having $A \cap B$ as its mathematical representative. It is important to note that this cannot be taken for granted. To understand the meaning of the Postulate recall the concept of an experimental proposition. An experimental proposition presupposes an observation space which in turn presupposes a (finite) set of compatible observables, in the simplest case one observable. Note that these are purely physical notions. What the *Postulate* says is this. Given two observation spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ and two experimental propositions P_1 and P_2 respectively which are mathemtically repesented by the closed subspaces A and B respectively. Then it is *postulated* that there exists an observation space $\mathcal{S}_3$ and an experimental proposition P_3 relative to $\mathcal{S}_3$ such that the subspace $A \cap B$ is the mathematical representation of P_3. So the Postulate is about possible observation spaces and thus about possible observables. It may be viewed as a physical postulate or at least as a postulate concerning the link between physical observables and the logic and the formalism of quantum mechanics. It is in this light that the ensuing remark is to be understood: "This postulate would clearly be implied by the not unusual conjecture that all Hermitian-symmetric operators in Hilbert space (phase space) correspond to observables"

The reasoning now naturally proceeds as follows. Since the closed linear sum of any two closed subspaces A and B, i.e. $A \wedge B$ is $A \cap B^{\perp}$ they conclude:

"The set product and closed linear sum of any two, and the orthogonal complement of any one closed linear subspace of Hilbert space representing mathematically an experimental proposition concernimg a quantum-mechanical system $\mathcal{S}$, is itself the representation of an experimental proposition concerning $\mathcal{S}$"

They continue: "...this defines the calculus of experimental propositions concerning $\mathcal{S}$, as a calculus of experimental propositions and a relation of implication..."

There is a passage in Paragraph 4 which also needs interpretation. "In quantum theory ...the possibility of predicting in general the readings from measurements on a physical system $\mathcal{S}$ from a knowledge of its 'state' is denied; only statistical predictions are always possible. This has been interpreted as a renunciation of the doctrine of pre-determination; a thoughtful analysis shows that another and more subtle idea is involved. The central idea is that physical quantities are *related*, but are not all computable from a number of *independent basic* quantities (such as position and velocity). We shall show in paragraph 12 that this situation has an exact algebraic analogue in the calculus of propositions."

What now is said in Paragraph 12? Here they say: "...we conclude that *the propositional calculus of quantum mechanics has the same structure as an abstract projective geometry.*"

Let us try an interpretation of this statement. In classical mechanics all physical quantities can be computed, as is the terminology of BvN, from certain basic quantities, namely position and velocity. If we substitute 'deducible' for 'computable' here, the above statement could mean that the 'calculus' of the

propositions of quantum mechanics is not a deductive system (calculus) as is classical logic. Rather it has the structure of a (projective) geometry describing *relations* between states and propositions analogous to the relations we have in a projective geometry between points, lines, hyperplanes... One might say that the quantum mechanical calculus is geometrical rather than deductive.

This is an idea which becomes fully transparent in the concept of Hilbert space logic which we will introduce later in the book.

Let us now give a short summary of part 2 of the paragraph entitled "Algebraic Analysis". As indicated in the title this part is more technical in nature and less heuristic than the first part.

First it is suggested that the calculus of experimental propositions should have an implication and this implication should be set inclusion.

The authors then proceed by defining the concept of a lattice. It is suggested that the experimental propositions should form a lattice and should thus satisfy the 'laws' that hold in any lattice such as commutativity of join and meet and also associativity.

They then define the concept of a complemented lattice remarking that in the case of closed subspaces of a Hilbert space complementation is orthogonal complement formation.

In paragraph 10 an important issue is discussed, namely the question whether the calculus of quantum mechanics should satisfy the distributive identity, in BvN's notation

$$L6 \ a \cup (b \cap c) = (a \cup b) \cap (a \cup c$$

as well as its dual form. It is argued that the calculus of quantum mechanics does not satisfy the distributive identity. This is even considered the 'central difference' between the logic of classical and quantum mechanics. They write: "It is interesting that L6 is also a logical consequence of the compatibility of the observables occurring in a and b and c... These facts suggest that the distributive law *may* break down in quantum mechanics"

Note that this passage is somewhat vague. What does the phrase "compatible observables occurring in ..." mean? Again, it seems that this argument is heurisic Obviously BvN do not yet have the concept of compatibility of elements in a lattice, see ? It is, however, suggested that a weakened form of the distributive law should hold, namely the following identity called *modularity*, see ?, in Bvn 's notation

$$L5 \text{ If } a \subset b, \text{ then } a \cup (b \cap c) = (acupb) \cap c$$

It is pointed out that finite-dimensional subspaces of a Hilbert space do satisfy the modular identity. Moreover, it is shown by a counterexample this is not the case for infinite -dimensional closed subspaces, see also ?

Birkhoff and von Neumann make an interesting mathematical observation in that they observe that the modular identity follows from the existence of a 'numerical dimension function' d, i.e. a function with the following properties.

$$D1\text{: If } a > b, \text{ then } d(a) > d(b)$$

and

$$D2\colon\ d(a) + d(b) = d(a \cap b) + d(a \cup b)$$

Note that the modular identity is stronger than the orthomodular identity, see ?. It is interesting that BvN insist on this stronger version which, as already noted, does not necessarily hold for infinite -dimensional closed subspaces of a Hilbert space. As pointed out by Redei in [?] it is von Neumann's hope that such a dimension function which is similar to a probability function might be of use in understanding the probabilistic nature of Quantum Mechanics.

In many publications the Birkhoff-von Neumann paper is quoted as introducing the lattice of closed subspaces of a Hilbert space as the 'logic of quantum mechanics'. This statement appears unfounded in view of what Birkhoff and von Neumann say in "Conclusions": "One conclusion one can draw from the preceding algebraic considerations, is that one can construct many different models for a propositional calculus of quantum mechanics, which cannot be distiguished by known criteria". What follows then says in modern terminology that any finite dimensional projective geometry over any field with a suitable involuntary anti-automorphism and a Hermitian form. It is worth noting that they insist on finite dimensionality because they want to retain the modular law. Even in the infinite-dimensional case they consider it -in the spirit of Hankel's principle "perseverance of formal laws"- desirable to retain the modular law. But this excludes the closed subspaces of a finite-dimensional Hilbert space as a proper 'infinite-dimensional model' of the logical calculus of quantum mechanics. It is this fact that led von Neumann to develop his "Continuous Geometry".

A letter by von Neumann to Birkhoff written before the paper was published contains the following passage (from Redei [55]: "Your general remarks, I think, are very true: I too think that our paper will not be exhaustive or conclusive, but that we should not attempt to make it such : The subject is obviously only at the beginning of a development, and we want to suggest the direction of this development much more, than reach 'final results'. I, for one, do not even believe, that the right formal frame for quantum mechanics is already found."

5.0.5 The correspondence between Birkhoff and von Neumannn during the writing of the paper

von Neumann to Birkhoff

The following has already been said.

"Your general remarks I think are very true: I, too, think that our paper will not be very exhaustive or conclusive, but that we should not attempt to make it such: The subject is obviously only at the beginning of a development, and we want to suggest the direction of this development much more, than to reach final results. I, for one, do not even believe that the right formal frame for quantum mechanics is already found."

Birkhoff' s conditional

The following should probably be said in connection with Implication M-algebras.

von Neumann to Birkhoff: "Last spring you observed: Why not introduce a logical operation *ab* for any two (not necessarily simultaneously decidable) properties a and b, meaning this: If you first measure a you find that it is present, if you next measure b, you find that it is present too.

This *ab* cannot be described by any operator, and in particular not by a projection (= linear subspace). The only answer I could then find was this: There is no state in which the property *ab* is certainly present, nor is any in which it is certainly absent (assuming that a,b are sufficiently non-simultaneously definable = that their projection operators E,F have no common proper-functions not = 0 at all).

Of course, for this reason *ab* is no physical quantity relatively to the the machinery of quantum -mechanics. But how can one motivate this, how can one find a criterion of what is a physical quality and what not, if not by the 'causality' criterion: A statement describes if and only if the states in which it can be decided with certainty form a complete set.

I wanted to avoid this rather touchy and complicated question, and withdraw to the safe - although perhaps narrow- position of dealing with 'causal' statements' only. Do you propose to discuss the question fully? It might become too philosophical, but I would not say that I object absolutely to it. But it is dangerous ground-except you have a new idea , which settles the question more satisfactorily."

Let us take a closer look at this letter. In this letters JvN refers to a 'logical operation' proposed by Birkhoff in an earlier letter (spring 1935?). What does Birkhoff mean by 'logical operation'? A oonnective probably, so "ab" is a conditional in our sense. The conditional proposed by Birkhoff is similar to but different from ours. Our conditional says: "If you measure A, you end up in a state in which B is sharp." Think of "sharp" as "provable". So given a state x. Then after masuring A we are in state Ax. The requirement that in this state B is sharp says that $BAx = Ax$. In this terminology, Birkhoff's conditional says: "If you first measure A and then measure B, you end up in a state in which A is still sharp.". Mathematically this means: $ABAx = BAx$.

Now JvN is not satisfied with this arguing that this 'operation' is not representable by a projection (subspace). The following is a reconstruction of his argument. Let A, B such that $A \cup B = \{0\}$, not $B \subset A^\perp$ and not $B \subset A$. Such a constellation exists in every Hilbert space of dimension at least 3.

His first observation is : "There is no state in which the property ab is certainly present" What does this mean? For A, B as above, $ABAx = BAx$ implies $BAx = 0$ or equivalently $Ax \in B^\perp$

In our terminology, the consequence relation has no internalising connective definable by the lattice operations.

Think about this again.

We will come back to this Chapter later in "Birkhoff-von Neumann Revisited", where we we will establish the precise connection betweeen Birkhoff-von

Neumann and the approach put forward in this book

5.0.6 The Kochen-Specker and the Schütte Tautologies

Although Birkhoff and von Neumann must have been fully aware of the fact that their logical 'calculus' must differ from classical logic in important respects, they did not provide an explicit comparison between the two logics. The only difference they explicitly mention is that the distributive laws of classical logic no longer hold in the 'logic of quantum mechanics'. In view of the fact that it was their chief motivation to incorporate the profound differences between classical and quantum mechanics such as the existence of uncertainty relations into the logic this seems not too profound an observation. It took some time until a truly surprising phenomenon was found which highlights the profound difference between Birkhoff-von Neumann quantum logic and classical logic. Namely, Kochen and Specker discovered in their classic paper "The problem of hidden variables in quantum mechanics" [33], as a byproduct, the following fact. There exists a classical tautology which under a certain 'valuation' in the lattice of closed subspaces in (three-dimensional) Hilbert space represents the zero space. Loosely speaking there exists a classical tautology which is a 'quantum logicical contradiction' and the other way round. Kochen and Specker explicitly present such a tautology in 117 propositional variables. A similar tautology had even before the publication of the Kochen-Specker paper been found by Schütte as is known from a letter that Schütte wrote to Specker. Schütte's tautology does not represent the zero space. But it does not represent the whole space either. It is a classical tautology which is not a quantum tautology. It is important to note that both in the Kochen-Specker tautology and in Schütte's tautology only compatible quantum propositions are combined via the connectives.

We will study this phenomenon of classical inconsistency in the Birkhoff-von Neumann quantum logic from a general point of view in Chapters 7 and 9. We will see that this phenomenon is by no means accidental.

Chapter 6

The Dynamic Viewpoint: Propositions as Operators

6.1 Propositions viewed dynamically

Let us begin by pointing out a certain analogy between measurements and propositions. A physical measurement, e.g., measuring the temperature of a gas to be $138\,^{\circ}K$, asserts that the proposition *the temperature of this gas is* $138\,^{\circ}K$ holds true. A measurement, in a sense, asserts the truth of a proposition. This is the fundamental analogy between physics and logic: making a measurement is similar to asserting a certain kind of proposition. The example above has been taken from classical physics. Consider now measuring the spin of a particle along the z-axis to be $1/2$. This measurement is akin to asserting the truth of the proposition *the spin along the z-axis is* $1/2$. But, here, the assertion of the proposition, i.e., the measurement, changes the state of the system. The assertion holds in the state resulting from the measurement, but did not necessarily hold in the state of the system before the measurement was performed. In fact it held in this previous state if and only if the measurement left the state unchanged. Inspired by the analogy between measurements and propositions we set ourselves to study the logic of propositions that not only *hold* at states, i.e., models, but also *act* on them, transforming the state in which they are evaluated into another one. A proposition holds in some state if and only if this state is a fixpoint for the proposition.

Conceptualy this is the novelty of our approach to logic in this chapter. We view propositions in a dynamic rather than in a static way. The motivation for this is provided by the analogy between measurements and propositions, and thus in this chapter we restrict ourselves to propositions having a physical meaning such as "The energy of the electron in the hydrogen atom is such and such". One can now ask the question whether this dynamic aspect of propositions is peculiar to a cetrain type of propositions, namely quantum mechanical propositions, or whether we should consider the dynamic view of propositions as

the proper one even beyond the realm of quantum mechanics. Let us just note that in the technical treatment of the dynamic nature of propositions presented in this chapter the static view characteristic of traditional, in particular classical logic is a special case of the dynamic view. We will go into this in more detail in the last chapter of this book.

6.2 The Concept of an M-Algebra

In inpired by the above consideration we define a new class of abstract structures for which we coin the term *algebras of measurements*, M-algebras for short. Formally, this concept is an abstraction from the the algebra of projections of a Hilbert space. The most appealing feature of these structures consists in the intuitive content of its axioms. The main intuition is, as already pointed out, the analogy between measurements and propositions. Projections in Hilbert space may be viewed as measurements that can change the state of the system as is the case in quantum mechanics.

Section **??** will summarize in a most succinct and formal way the definition of Algebras of Measurements (M-algebras), by presenting a list of properties. It should be used as an overview and memento only. The following sections will explain the properties, present motivation and explanation, and then prove basic properties of M-algebras.

The structures we are concerned with deal with a set X and functions from X to X. We shall denote the composition of functions by $\circ$ and composition has to be understood from left to right: for any $x \in X$, $(\alpha \circ \beta)(x) = \beta(\alpha(x))$. If $\alpha : X \longrightarrow X$, we shall denote by $FP(\alpha)$ the set of all fixpoints of α: $FP(\alpha) \stackrel{\text{def}}{=} \{x \in X \mid \alpha(x) = x\}$.

Definition 6.1 *An M-algebra is a pair $\langle X, M\rangle$ in which X is a non-empty set and M is a set of functions from X to X, that satisfies the six properties described below.*

- **Illegitimate** $\exists\, 0 \in X$ such that $\forall \alpha \in M$, $0 \in FP(\alpha)$, i.e, $\alpha(0) = 0$.
- **Idempotence** $\forall \alpha \in M$, $\alpha \circ \alpha = \alpha$, i.e., for any $x \in X$, $\alpha(\alpha(x)) = \alpha(x)$.

  The next property requires a preliminary definition.

  Definition 6.2 *For any $\alpha, \beta : X \longrightarrow X$, we shall say that α* preserves *β if and only if α preserves $FP(\beta)$, i.e., if $\alpha(FP(\beta)) \subseteq FP(\beta)$, i.e., $\forall x \in X$, $\beta(x) = x \Rightarrow \beta(\alpha(x)) = \alpha(x)$.*

- **Composition** $\forall \alpha, \beta \in M$, if α preserves β, then $\beta \circ \alpha \in M$.
- **Interference** $\forall x \in X, \forall \alpha, \beta \in M$, if $x \in FP(\alpha)$, i.e., $\alpha(x) = x$, and $(\beta \circ \alpha)(x) \in FP(\beta)$, i.e., $\beta(\alpha(\beta(x))) = \alpha(\beta(x))$, then $\beta(x) \in FP(\alpha)$, i.e., $\alpha(\beta(x)) = \beta(x)$.

- **Cumulativity** $\forall x \in X$, $\forall \alpha, \beta \in M$, if $\alpha(x) \in FP(\beta)$, i.e., $\beta(\alpha(x)) = \alpha(x)$ and $\beta(x) \in FP(\alpha)$, i.e., $\alpha(\beta(x)) = \beta(x)$, then $\alpha(x) = \beta(x)$.

  The next property requires some notation. For any $\alpha : X \longrightarrow X$, we shall denote by $Z(\alpha)$ the set of zeros of α: $Z(\alpha) \stackrel{\text{def}}{=} \{x \in X \mid \alpha(x) = 0\}$.

- **Negation** $\forall \alpha \in M$, $\exists(\neg\alpha) \in M$, such that $FP(\neg\alpha) = Z(\alpha)$, and $Z(\neg\alpha) = FP(\alpha)$, i.e., $\forall x \in X,\ \alpha(x) = 0$ iff $(\neg\alpha)(x) = x$ and $\forall x \in X,\ \alpha(x) = x$ iff $(\neg\alpha)(x) = 0$.

An additional property will be considered in Section 6.8.

Definition 6.3 *An M-algebra is* separable *if it satisfies the following:*

Separability *For any $x, y \in X - \{0\}$, if $x \neq y$ then $\exists \alpha \in M$ such that $\alpha(x) = x$ and $\alpha(y) \neq y$.*

The above definition needs comment. The mathematically educated reader will easily realise that the structures called M-algebras defined above although do, in the strict sense of Universal Algebra, not qualify as algebras. From the point of view of Universal Algebra as well as from the viewpoint of Model Theory, algebras are first-order structures whereas what we call an M-algebra is a second order structure very much akin to a topological space. Perhaps the appropriate term to denote them should be "propositional space", a non-empty set X equipped with a set M of functions from X to X as a topological space is a non-empty set X equipped with a set of subsets of X. Both the elements of M in the case of an M-algebra and the topology in the case of a topological space are second order entities. The axiomas of both an M-algebra and a topological space are formulated in a second order language quantifying over second prder entities. We retain the term "M-algebra" for the sake of a suggestive terminology reminding the reader of the analogy with measurements in quantum physics.

6.3 Motivation and Justification

In this section, we shall leisurely explain each one of the properties described in Section **??**. Our explanation of each property will include three parts:

- an epistemological explanation whose purpose is to explain why the property is natural or even required when one thinks of measurements,
- an explanation of why the property holds in the algebra $\langle H, L\rangle$ where H is a Hilbert space and L the set of all projections onto closed subspaces of H,
- an explanation of the logical meaning of the property, based on the identification of measurements with propositions.

6.3.1 States

We shall reserve the term *state* for the elements of X. In physical terms, the set X is the set of all possible states of a system. When we say *state* we mean a state as fully determined as is physically possible: e.g., in classical mechanics, a set of $6n$ values if we consider n particles (three values for position and three values for momentum), or what is generally termed, in Quantum Physics a *pure state*.

In the Hilbertian description of Quantum Physics, a (pure) state is a one-dimensional subspace, i.e., a ray, in some Hilbert space. The illegitimate state, 0 is the zero-dimensional subspace.

In our study of M-algebras a state is a primitive notion and we need not reflect in depth on this at this stage. We will enter a detailed discussion of the concept of state in Chapter 7.

6.3.2 Measurements

The elements of M represent measurements on the physical system whose possible states are those of X. In Classical Physics one may assume that a measurement leaves the measured system unchanged. It is a hallmark of Quantum Physics that this assumption cannot be held true anymore. In Quantum Physics, measurements, in general, change the state of the system. This is the phenomenon called *collapse of the wave function*. Therefore we model measurements by transformations on the set of states. Clearly not any transformation can be called a measurement. A measurement changes the system in some minimal way. A transformation that brings about a wild change in the system cannot be considered to be a measurement. Many of the properties presented above and discussed below explicit this requirement.

A word of caution is necessary here before we proceed. When we speak about measurement we do not mean some declaration of intentions such as *measuring the position of a particle*, we mean the action of measuring some physical quantity *and finding a specific value*, such as *finding the particle at the origin of the system of coordinates*. Measuring $0.3\,^{\circ}K$ and measuring $1000\,^{\circ}K$ are not two different possible results for the same measurement, they are two different measurements.

In the Hilbertian description of Quantum Physics measurable quantities are by Hermitian operators. Measurements in our sense are represented by a pair $\langle A, \lambda \rangle$ where A is a Hermitian operator and λ an eigenvalue of A. The effect of measuring $\langle A, \lambda \rangle$ in state x is to project x onto the eigensubspace of A for eigenvalue λ. A measurement α is therefore a projection on a closed subspace of a Hilbert space. The set $FP(\alpha)$ is the closed subspace on which α projects. Those projections onto eigensubspaces are the measurements we try to identify. Our goal is to identify the algebraic properties of such projections that make them suitable to represent physical measurements in Quantum Physics.

¿From a classical logician's point of view, a measurement is a proposition. A proposition α acts on a state, i.e., a theory T by sending it to the theory that

results from adding α to T and then closing under logical consequence. One sees that, from this point of view, if T is maximal then $\alpha(T)$ is either T (iff α is in T) or the inconsistent theory. We see here that a proposition (measurement) holds in some model (state) if and only if the model is a fixpoint of the proposition.

This is the interpretation that we shall take along with us: a measurement α *holds* at some state x, or, equivalently x satisfies α, if and only if $x \in FP(\alpha)$.

6.3.3 Illegitimate

Illegitimate is mainly a technical requirement. The sequel will show why it is handy. The illegitimate state 0 is a state that is physically impossible. Physicists, in general, do not consider this state explicitly, we shall. ¿From the epistemological point of view, we just require that amongst all the possible states of the system we include a state, denoted 0 that represents physical impossibility. There is not much sense in measuring anything in the illegitimate state, therefore, it is natural to assume that no measurement α operating on the illegitimate state can change it into some legitimate state. This is the meaning of our requirement that 0 be a fixpoint of any measurement. In other terms, the state 0 satisfies every measurement, every measurement holds at 0.

In the Hilbertian description of Quantum Physics the zero vector plays the role of our 0. Indeed, since a projection is linear, it preserves the zero vector.

¿From a logician's point of view **Illegitimate** requires us to include the inconsistent theory in X. Clearly, the result of adding any proposition to the inconsistent theory leaves us with the inconsistent theory.

6.3.4 Zeros

We have described in Section 6.3.2 the interpretation we give to the fact that a state x is a fixpoint of a measurement α. We want to give a similarly central meaning to the fact that a state x is a zero of a measurement α: $x \in Z(\alpha)$, i.e., $\alpha(x) = 0$. If measuring α sends x to the illegitimate state, measuring α is physically impossible at x. This should be understood as meaning that the state x has some definite value different from the one specified by α.

If, at x, the spin is $1/2$ along the z-axis, then measuring along the z-axis a spin of $-1/2$ is physically impossible and therefore the measurement of $-1/2$ sends the state x to the illegitimate state 0. The status of the measurement that measures $-1/2$ along the *x-axis* is completely different: this measurement does not send x to 0, but to some legitimate state in which the spin along the x-axis is $-1/2$.

It is natural to say that a measurement α has a definite value at x iff x is either a fixpoint or a zero of α. We shall define: $Def(\alpha) \stackrel{\text{def}}{=} FP(\alpha) \cup Z(\alpha)$. If $x \in Def(\alpha)$, α has a definite value at x: either it holds at x or it is impossible at x. If $x \notin Def(\alpha)$, $\alpha(x)$ is some state different from x and different from 0.

In the Hilbertian presentation of Quantum Physics, the zeros of a measurement α are the vectors orthogonal to the set of fixpoints of α.

6.3.5 Idempotence

Idempotence is extremely meaningful. It is an epistemologically fundamental property of measurements that they are idempotent: if α is a measurement and x a state, then $\alpha(\alpha(x)) = \alpha(x)$, i.e., measuring the same value twice in a row is exactly like measuring it once. Note that, by **Illegitimate**, if $x \in Def(\alpha)$, then $\alpha(\alpha(x)) = \alpha(x)$. The import of **Idempotence** concerns states that are not in $Def(\alpha)$.

It seems very difficult to imagine a scientific theory in which measurements are not idempotent: it would be impossible to check directly that a system is indeed in the state we expect it to be in without changing it. Idempotence is one of the conditions that ensure that measurements change states only minimally. This principle seems to be a fundamental principle of all science, having to do with the reproducibility of experiments. If there was a physical system and a measurement that, if performed twice in a row gave different results, then such a measurement would be, in principle, irreproducible.

In the Hilbertian description of Quantum Physics measurements are modeled by projections onto eigensubspaces. Any projection is idempotent. But it is enlightening to reflect on the phenomenology of this idempotence. For an electron whose spin is positive along the z-axis (state x_0), measuring a negative spin along the x-axis is feasible, i.e., does not send the system into the illegitimate state, but sends the system into a state (x_1) different from the original one, x_0. Nevertheless, a consequence of the collapse of the wave function is that, after measuring a negative spin along the x-axis, the spin is indeed negative along the x-axis and therefore a new measurement of a negative spin along the x-axis leaves the state x_1 of our electron unchanged, whereas measuring a positive spin along the x-axis is now an unfeasible measurement and sends x_1 to the illegitimate state. Note that such a measurement of a positive spin along the x-axis in the original state x_0 brings us to a legitimate state x_3 different from x_0 and x_1. The idempotence of measurements, probably epistemologically necessary, provides some explanation of why projections in Hilbert spaces are a suitable model.

¿From the logical point of view, idempotence corresponds to the fact that asserting the truth of a proposition is equivalent to asserting it twice. For any reasonable consequence operation $\mathcal{C}$

$\mathcal{C}(\mathcal{C}(T, a), a) = \mathcal{C}(T, a)$.

6.3.6 Preservation

The definition of *preservation* encapsulates the way in which different measurements can interfere. If α preserves $FP(\beta)$, the set of states in which β holds, α never destroys the truth of proposition β: it never interferes badly with β.

6.3.7 Composition

Composition has physical significance. It is a global principle: it assumes a global property and concludes a global property. Measurements are mappings of X into itself, therefore we may consider the composition of two measurements. According to the principle of minimal change, we do not expect the composition of two measurements to be a measurement: two small changes may make a big change. But, if those two measurements do not interfere in any negative way with each other, we may consider their composition as small changes that do not add up to a big change. **Composition** requires that if, indeed, α preserves β, then the composite operation that consists of measuring β first, and then α does not add up to a big change and should be a bona fide measurement. Notice that we perform β first, whose result is (by **Idempotence**) a state that satisfies β, then we perform α, which does not destroy the result obtained by the first measurement β.

In the Hilbertian presentation of Quantum Physics, consider α, the projection on some closed subspace A and β, the projection on B. The measurement α preserves β iff the projection of the subspace B onto A is contained in the intersection $A \cap B$ of A and B. In such a case the composition $\beta \circ \alpha$ of the two projections, first on B and then on A is equivalent to the projection on the intersection $A \cap B$. It is therefore a projection on some closed subspace.

For the classical logician, measurements always preserve each other. If $a \in T$, then $a \in \mathcal{C}(T, b)$ for any proposition b. This is a consequence of the monotonicity of $\mathcal{C}$. **Composition** requires that the composition of any two measurements be a measurement. For the logician, $\beta \circ \alpha$ is the measurement $\beta \wedge \alpha$. **Composition** amounts to the assumption that M is closed under conjunction.

Technically, the role of **Composition**is to ensure that two commuting measurements' composition is a measurement. Equivalently, we could have, instead of **Composition**, required that for any pair $\alpha, \beta \in M$ such that $\alpha \circ \beta = \beta \circ \alpha$, their composition $\alpha \circ \beta$ be in M.

6.3.8 Interference

Interference has a deep physical meaning. It is a local principle, i.e., holds separately at each state x. It may be seen as a local logical version of Heisenberg's uncertainty principle. It considers a state x that satisfies α. Measuring β at x may leave α undisturbed (this is the conclusion), but, if β disturbs α, then no state at which both α and β hold can ever be attained by measuring α and β in succession. In other words, either such a state, satisfying both α and β is obtained immediately, or never.

We shall say that β disturbs α at x if $x \in FP(\alpha)$ but $\beta(x) \notin FP(\alpha)$. Note that β preserves α if and only if it disturbs α at no x. **Interference** says that if β disturbs α at x then α disturbs β at $\beta(x)$, and β disturbs α at $(\beta \circ \alpha)(x)$, and so on. We chose to name this property *Interference* since it deals with the local interference of two measurements: if they interfere once, they will continue interfering ad infinitum.

In the Hilbertian presentation of Quantum Physics, the principle of **Interference** is satisfied for the following reason. Consider a vector $x \in H$ and two closed subspaces of H: A and B. Assume x is in A. Let y be the projection of x onto B and z the projection of y onto A. Assume that z is in B. Since both x and z are in A, the vector $z - x$ is in A. Similarly, the vector $z - y$ is in B. But y is the projection of x onto B and therefore $y - x$ is orthogonal to B and in particular orthogonal to $z - y$. We have $\langle y - x, z - y \rangle = 0$, and

$$\langle y, z \rangle - \langle y, y \rangle - \langle x, z \rangle + \langle x, y \rangle = 0.$$

Since z is the projection of y onto A, the vector $z - y$ is orthogonal to A and we have $\langle (z - x), (z - y) = 0 \rangle$, and

$$\langle z, z \rangle - \langle z, y \rangle - \langle x, z \rangle + \langle x, y \rangle = 0.$$

By substracting the first equality from the second we get:

$$-\langle z, y \rangle - \langle y, z \rangle + \langle y, y \rangle + \langle z, z \rangle = \langle y - z, y - z) \rangle = 0.$$

We conclude that $y = z$.

For the logician, it is always the case that $\beta(x) \in FP(\alpha)$ if $x \in FP(\alpha)$, as noticed in Section 6.3.7.

6.3.9 Cumulativity

Cumulativity is motivated by Logic. It does not seem to have been reflected upon by physicists. It parallels the cumulativity property that is central to nonmonotonic logic: see for example [**?**, **?**, **?**]. If the measurement of α at x causes β to hold (at $\alpha(x)$), and the measurement of β at x causes α to hold (at $\beta(x)$) then those two measurements have, locally (at x), the same effect. Indeed, they cannot be directly distinguished by testing α and β. **Cumulativity** says that they cannot be distinguished even indirectly.

In the Hilbertian formalism, if the projection, y, of x onto some closed subspace A is in B (closed subspace) then y is the projection of x onto the intersection $A \cap B$. If the projection z of x onto B is in A, z is the projection of x onto the intersection $B \cap A$ and therefore $y = z$. In fact, a stronger property than **Cumulativity** holds in Hilbert spaces. The following property, similar to the Loop property of [**?**], holds in Hilbert spaces: **L-Cumulativity** $\forall x \in X$, for any natural number n and for any sequence $\alpha_i \in M$, $i = 0, \ldots, n$ if, for any such i, $\alpha_i(x) \in FP(\alpha_{i+1})$, where $n + 1$ is understood as 0, then, for any $0 \leq i, j \leq n$, $\alpha_i(x) = \alpha_j(x)$.

To see that this property holds in Hilbert spaces, consider the distance d_i between x and the closed subspace A_i on which α_i projects. The condition $\alpha_i(x) \in FP(\alpha_{i+1})$ implies that $d_{i+1} \leq d_i$. We have $d_0 \geq d_1 \geq \ldots \geq d_n \geq d_0$ and we conclude that all those distances are equal and therefore $\alpha_i(x) \in FP(\alpha_{i+1})$ implies that $\alpha_i(x) = \alpha_{i+1}(x)$. We do not know whether the stronger **L-Cumulativity** is meaningful for Quantum Physics, or simply an uninteresting consequence of the Hilbertian formalism.

For the logical point of view, one easily sees that any classical measurements satisfy **Cumulativity**, and even **L-Cumulativity**.

6.3.10 Negation

Negation also originates in Logic. It corresponds to the assumption that propositions are closed under negation. If α is a measurement, α tests whether a certain physical quantity has a specific value v. If such a test can be performed, it seems that a similar test could be performed to test the fact that the physical quantity of interest has some other specific value or does not have value v.

In the Hilbertian formalism, to any closed subspace corresponds its orthogonal subspace, also closed.

For the logician, **Negation** amounts to the closure of the set of (classical) measurements, i.e., formulas, under negation.

6.3.11 Separability

We remind the reader that **Separability** is not included in the defining properties of an M-algebra. **Separability** asserts that if any two non-zero states x and y are different, there is a measurement that holds at x and not at y. Indeed, if all measurements that hold at x also hold at y it would not be possible to be sure that the system is in x and not in y. Compared to the previous requirements, **Separability** is of quite a different kind. It is some akin to a superselection principle, though presented in a dual way: a restriction on the set of states not on the set of observables.

Note that this implies that, in any non-trivial M-algebra (an M-algebra is trivial if $X = \{0\}$ and $M = \emptyset$), every state satisfies some measurement.

In the Hilbertian formalism, the projections on the one-dimensional subspaces defined by x and y respectively do the job.

For the logician, if T_1 and T_2 are two maximal consistent sets that are different, there is a formula α in $T_1 - T_2$. But, one may easily find (non-maximal) different theories T_1 and T_2 such that $T_1 \subset T_2$, contradicting **Separability**.

6.4 Examples of M-algebras

In this section we shall formally define the two paradigmatical examples of M-algebras that have been described in Section 6.3: propositional calculus and Hilbert space.

6.4.1 Logical Examples

Propositional Calculus: a non-separable M-algebra and a separable one

We shall now formalize our treatment of Propositional Calculus as an M-algebra. In doing so, we shall present Propositional Calculus in the way advocated by

Tarski and Gentzen. Let $\mathcal{L}$ be any language closed under a unary connective $\neg$ and a binary connective $\wedge$. Let $\mathcal{C}n$ be any consequence operation satisfying the following conditions (the conditions are satisfied by Propositional Calculus).

$$\textbf{Inclusion } \forall A \subseteq \mathcal{L},\ A \subseteq \mathcal{C}n(A),$$

$$\textbf{Monotonicity } \forall A, B \subseteq \mathcal{L}, A \subseteq B \ \Rightarrow\ \mathcal{C}n(A) \subseteq \mathcal{C}n(B),$$

$$\textbf{Idempotence } \forall A \subseteq \mathcal{L}, \mathcal{C}n(A) = \mathcal{C}n(\mathcal{C}n(A)),$$

$$\textbf{Negation } \forall A \subseteq \mathcal{L}, a \in \mathcal{L}, \mathcal{C}n(A, \neg a) = \mathcal{L} \ \Leftrightarrow\ a \in \mathcal{C}(A),$$

$$\textbf{Conjunction } \forall A \subseteq \mathcal{L}, a, b \in \mathcal{L},\ \mathcal{C}n(A, a, b) = \mathcal{C}n(A, a \wedge b).$$

Define a subset of $\mathcal{L}$ to be a *theory* iff it is closed under $\mathcal{C}n$: $T \subseteq \mathcal{L}$ is a theory iff $\mathcal{C}n(T) = T$. Let X be the set of all theories. Let M be the language $\mathcal{L}$. The action of a formula $\alpha \in \mathcal{L}$ on a theory T is defined by: $\alpha(T) = \mathcal{C}n(T \cup \{\alpha\})$. In such a structure α holds at T iff $\alpha \in T$. Let us check that such a structure satisfies all the defining properties of an M-algebra. We shall not mention the uses of Inclusion. The illegitimate state is the theory $\mathcal{L}$. **Idempotence** follows from the property of the same name. **Composition** follows from **Conjunction**: the composition $a \circ b$ is the measurement $a \wedge b$. Note that any pair of measurements commute. **Interference** is satisfied because $a \in T$ implies $a \in \mathcal{C}n(T, b)$. **Cumulativity** is satisfied because $b \in \mathcal{C}n(T, a)$ implies $\mathcal{C}n(T, a) = \mathcal{C}n(T, a, b)$ by **Monotonicity** and **Idempotence**. **Negation** holds by the property of the same name.

The M-algebra above does not satisfy **Separability** since there are theories T and S such that $T \subset S$ and every formula α satisfied by T is also satisfied by S. This M-algebra is *commutative*: any two measurements commute since: $\mathcal{C}n(\mathcal{C}(T, a), b) = \mathcal{C}n(\mathcal{C}(T, b), a)$.

If we consider the subset $Y \subset X$ consisting only of *maximal consistent* theories and the inconsistent theory, we see that the pair $\langle Y, \mathcal{L} \rangle$ is an M-algebra, because Y is closed under the measurements in $\mathcal{L}$. In this M-algebra, all measurements do more than commute, they are *classical*, in the following sense.

Definition 6.4 *A mapping $\alpha : X \longrightarrow X$ is said to be* classical *iff for every $x \in X$, either $\alpha(x) = x$ or $\alpha(x) = 0$.*

The M-algebra above is separable: if T_1 and T_2 are diferent maximal consistent theories there is a formula $a \in T_1 - T_2$.

Nonmonotonic inference operations

In Section 6.4.1 we assumed that the inference operation $\mathcal{C}n$ was monotonic. It seems attractive to consider the more general case of nonmonotonic inference operations studied, for example in [**?**]. More precisely what about replacing **Monotonicity** by the weaker

$$\textbf{Cumulativity } \forall A, B \subseteq \mathcal{L},\ A \subseteq B \subseteq \mathcal{C}(A) \ \Rightarrow\ \mathcal{C}(B) = \mathcal{C}(A).$$

Notice that, in such a case, we prefer to denote our inference operation by $\mathcal{C}$ and not by $\mathcal{C}n$. The reader may verify that all requirements for an M-algebra still hold true, *except for* **Composition**. In such a structure all measurements still commute and we therefore need that every composition $a \circ b$ of measurements (formula) be a measurement (formula). But the reader may check that $a \wedge b$ does not have the required properties: $\mathcal{C}(T, a \wedge b) = \mathcal{C}(T, a, b)$ but, since $\mathcal{C}$ is not required to be monotonic, there may well be some formula $c \in \mathcal{C}(T, a)$ that is not in $\mathcal{C}(T, a, b)$. In such a case $\mathcal{C}(T, a \wedge b) \neq \mathcal{C}(\mathcal{C}(T, a), b)$, as would be required. One may, then, think of extending the language $\mathcal{L}$ to include formulas of the form box$a \circ b$ acting as compositions. But the **Negation** condition of the definition of an M-algebra requires every formula (measurement) to have a negation and there is no obvious definition for the negation of a composition. The **Monotonicity** property seems therefore essential.

Revisions

Another natural idea is to consider revisions a la AGM [**?**]. The action of a formula a on a theory T would be defined as the theory T revised by a: $T{*}a$. The structure obtained does not satisfy the M-algebra assumptions. The most blatant violation concerns **Negation**. In revision theory negation does not behave at all as expected in an M-algebra.

6.4.2 Orthomodular and Hilbert spaces

Orthomodular spaces

Given any orthomodular space $\mathcal{H}$, denote by M the set of all closed subspaces of $\mathcal{H}$. Then the pair $\langle \mathcal{H}, M \rangle$ is an M-algebra, if any $\alpha \in M$ acts on $\mathcal{H}$ in the following way: $\alpha(x)$ is the unique vector such that $x = \alpha(x) + y$ for some vector $y \in \alpha^{\perp}$. In light of Section 6.3 the reader will have no trouble proving that any such structure is an M-algebra. It is not separable, though: any two colinear vectors satisfy exactly the same measurements. The next section will present a related separable M-algebra.

Rays

Given any orthomodular space $\mathcal{H}$, let X be the set of one-dimensional or zero-dimensional subspaces of $\mathcal{H}$. Let M be the set of closed subspaces of $\mathcal{H}$. The projection on a closed subspace is linear and therefore sends a one-dimensional subspace to a one-dimensional or a zero-dimensional subspace and sends the zero-dimensional subspace to itself. The pair $\langle X, M \rangle$ is easily seen to be an M-algebra. This M-algebra is separable: notice that $X \subset M$ and that $x \in X$ is the only state satisfying the measurement x.

6.5 Properties of M-algebras

We assume that $\langle X, M\rangle$ is an arbitrary M-algebra. First, we shall show that any M-algebra includes two trivial measurements: $\top$, analogous to the truth-value *true*, that leaves every state unchanged and measures a property satisfied by every state and $\perp$, analogous to *false*, that sends every state to the illegitimate state, and is nowhere satisfied.

Lemma 6.1 *[Negation, Composition, Idempotence] There are measurements $\top, \perp \in M$ such that for every $x \in X$, $\top(x) = x$ and $\perp(x) = 0$.*

Proof. The set M of measurements is not empty: assume $\alpha \in M$. Clearly, by **Negation**, the measurement $\neg\alpha$ preserves α. It follows, by **Composition**, that $\alpha \circ (\neg\alpha)$ is a measurement. Let $\perp = \alpha \circ (\neg\alpha)$. By **Idempotence** and **Negation**, for every $x \in X$, $\perp(x) = 0$. We now let $\top = \neg\perp$. ▮

Then, we want to show that measurements are uniquely specified by their fixpoints.

Lemma 6.2 *[Idempotence, Cumulativity] For any $\alpha, \beta \in M$, if $FP(\alpha) = FP(\beta)$, then $\alpha = \beta$.*

Proof. Assume $FP(\alpha) = FP(\beta)$. Let $x \in X$. By **Idempotence** $\alpha(x) \in FP(\alpha)$ and therefore, by assumption $\alpha(x) \in FP(\beta)$. Similarly $\beta(x) \in FP(\alpha)$. By **Cumulativity**, then, $\alpha = \beta$. ▮

Corollary 6.1 *[Idempotence, Cumulativity, Negation] For any $\alpha \in M$, $\neg\neg\alpha = \alpha$.*

Proof. Both α and $\neg\neg\alpha$ are measurements and $FP(\neg\neg\alpha) = FP(\alpha)$. ▮

We shall now prove a very important property. Suppose x is a state in which some measurement (i.e., proposition) holds: for example, at x the spin along the x-axis is $1/2$. Performing a measurement α on x may lead to a different state $y = \alpha(x)$. At y the spin along the x-axis may still be $1/2$, or it may be the case that the measurement α has interfered with the value of the spin. But, under no circumstance, can it be the case that the spin along the x-axis has a definite value different from $1/2$, such as $-1/2$. If the value of the spin along the x-axis at y is not $1/2$, the spin must be indefinite. This expresses the fact that a measurement α, acting on a state in which β holds, can either preserve β (when $\alpha(x) \in FP(\beta)$) or can disturb β (when $\alpha(x) \notin Def(\beta)$) but cannot make β impossible at x, i.e., $\alpha(x) \in Z(\beta)$. This is a very natural requirement stemming from the *minimal change* principle. A move from a definite value to a different definite value is too drastic to be accepted as measurement.

In the Hilbertian presentation of Quantum Physics, measurements are projections. The projection of a non-null vector x onto a closed subspace A is never orthogonal to x, unless x is orthogonal to A. Therefore if x is in some subspace B, but its projection on A is orthogonal to B, then this projection is the null vector.

Lemma 6.3 *[Illegitimate, Interference] For any $x \in X$, $\alpha, \beta \in M$, if $x \in FP(\beta)$, i.e., $\beta(x) = x$, and $\alpha(x) \in Z(\beta)$, i.e., $\beta(\alpha(x)) = 0$, then $x \in Z(\alpha)$, i.e., $\alpha(x) = 0$.*

Proof. Assume $x \in FP(\beta)$ and $\beta(\alpha(x)) = 0$. Then $(\alpha \circ \beta)(x) = 0 \in FP(\alpha)$. By **Interference**, then, $\alpha(x) \in FP(\beta)$ and $\beta(\alpha(x)) = \alpha(x)$, i.e., $0 = \alpha(x)$. ▮

We shall now sort out the relation between fixpoints and zeros. The next result is a dual of Lemma 6.3.

Lemma 6.4 *[Illegitimate, Interference, Negation] $\forall x \in X$, $\forall \alpha, \beta \in M$, if $x \in Z(\beta)$ and $\alpha(x) \in FP(\beta)$, then $x \in Z(\alpha)$. In other terms, if $\beta(x) = 0$ and $\beta(\alpha(x)) = \alpha(x)$, then $\alpha(x) = 0$.*

Proof. Consider the measurement $\neg\beta$ guaranteed by **Negation**. If we have $x \in FP(\neg\beta)$ and $\alpha(x) \in Z(\neg\beta)$, then, by Lemma 6.3 we have $x \in Z(\alpha)$. ▮

Lemma 6.5 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, $FP(\alpha) \subseteq FP(\beta)$ iff $Z(\beta) \subseteq Z(\alpha)$.*

Proof. Suppose $FP(\alpha) \subseteq FP(\beta)$ and $x \in Z(\beta)$. Since, by **Idempotence**, $\alpha(x) \in FP(\alpha)$, we have, by assumption, $\alpha(x) \in FP(\beta)$. By Lemma 6.4, then $x \in Z(\alpha)$.

Suppose now that $Z(\beta) \subseteq Z(\alpha)$. We have $FP(\neg\beta) \subseteq FP(\neg\alpha)$ and by what we just proved: $Z(\neg\alpha) \subseteq Z(\neg\beta)$. We conclude that $FP(\alpha) \subseteq FP(\beta)$. ▮

We shall now consider the composition of measurements. First we show the symmetry of the preservation relation.

Lemma 6.6 *[Idempotence, Interference] For any $\alpha, \beta \in M$, α preserves β iff β preserves α.*

Proof. Assume α preserves β, and $x \in FP(\alpha)$. By **Idempotence**, $\beta(x) \in FP(\beta)$. Since α preserves β, $\alpha(\beta(x)) \in FP(\beta)$. The assumptions of **Interference** are satisfied and we conclude that $\beta(x) \in FP(\alpha)$. We have shown that β preserves α. ▮

Lemma 6.7 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, if $\alpha \circ \beta \in M$, then $FP(\alpha \circ \beta) = FP(\alpha) \cap FP(\beta)$.*

Proof. Since $Z(\alpha) \subseteq Z(\alpha \circ \beta)$, Lemma 6.5 implies that $FP(\alpha \circ \beta) \subseteq FP(\alpha)$. By **Idempotence** of β, $FP(\alpha \circ \beta) \subseteq FP(\beta)$. We see that $FP(\alpha \circ \beta) \subseteq FP(\alpha) \cap FP(\beta)$. But the inclusion in the other direction is obvious. ▮

We shall now show that the converse of **Composition** holds.

Lemma 6.8 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, if $\alpha \circ \beta \in M$, then β preserves α.*

Proof. By Lemma 6.7, $FP(\alpha \circ \beta) \subseteq FP(\alpha)$. For any x, $(\alpha \circ \beta)(x)$ is therefore a fixpoint of α. Assume $x \in FP(\alpha)$. Then, $(\alpha \circ \beta)(x) = \beta(x)$ is a fixpoint of α. ∎

Lemma 6.9 *[Illegitimate, Idempotence, Interference, Composition, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$, iff β preserves α.*

Proof. The *only if* part is Lemma 6.8. The *if* part is **Composition**. ∎

Lemma 6.10 *[Illegitimate, Idempotence, Interference, Composition, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$ iff $\beta \circ \alpha \in M$.*

Proof. By Lemmas 6.9 and 6.6. ∎

Lemma 6.11 *[Illegitimate, Idempotence, Interference, Composition, Cumulativity, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$ iff α and β commute, i.e., $\alpha \circ \beta = \beta \circ \alpha$.*

Proof. Assume, first, that $\alpha \circ \beta \in M$. By Lemma 6.10, $\beta \circ \alpha \in M$. By Lemma 6.7, $FP(\alpha \circ \beta) = FP(\beta \circ \alpha)$, which implies the claim by Lemma 6.2.

Assume, now that α and β commute. We claim that α preserves β: indeed, if $\beta(x) = x$, then $\beta(\alpha(x)) = \alpha(\beta(x)) = \alpha(x)$ and therefore, by **Composition**, $\beta \circ \alpha$ is a measurement. ∎

Lemma 6.12 *[Illegitimate, Idempotence, Interference, Composition, Cumulativity, Negation] For any $\alpha, \beta \in M$, if $FP(\alpha) \subseteq FP(\beta)$, then $\alpha \circ \beta = \beta \circ \alpha = \alpha$.*

Proof. If $FP(\alpha) \subseteq FP(\beta)$, then, clearly $\alpha \circ \beta = \alpha$ by Idempotence of α. Therefore $\alpha \circ \beta \in M$ and, by Lemma 6.11, α and β commute. ∎

6.6 Connectives in M-algebras

6.6.1 Connectives for arbitrary measurements

The reader has noticed that *negation* plays a central role in our presentation of M-algebras, through the **Negation** requirement and that this requirement is central in the derivation of many of the lemmas of Section 6.5. Indeed, **Negation** expresses the orthogonality structure so fundamental in orthomodular and Hilbert spaces. The requirement of **Negation** corresponds, for the logician, to the existence of a connective whose properties are those of a classical negation. Indeed, for example, as shown by Corollary 6.1, double negations may be ignored, as is the case in classical logic. In [**?**], the logical language presented includes negation, interpreted as orthogonal complement, and this is consistent with our interpretation. But [**?**] also defines other connectives: conjunction, disjunction and many later works on Quantum Logic define also implication (sometimes a number of implications). Our treatment in this chapter does not

require such connectives, or more precisely, our treatment does not require that such connectives be defined between any pair of measurements.

Consider conjunction. One may consider only M-algebras in which, for any $\alpha, \beta \in M$ there is a measurement $\alpha \wedge \beta \in M$ such that $FP(\alpha \wedge \beta) = FP(\alpha) \cap FP(\beta)$. There are many such M-algebras, since any M-algebra defined by an orthomodular space and the family of all its (projections on) closed subspaces has this property since the intersection of any two closed subspaces is a closed subspace. But our requirements do not imply the existence of such a measurement $\alpha, \beta \in M$ for every α and β.

For disjunction, one may consider requiring that for any $\alpha, \beta \in M$ there be a measurement $\alpha \vee \beta \in M$ such that $Z(\alpha \vee \beta) = Z(\alpha) \cup Z(\beta)$, and the M-algebras defined by Hilbert spaces satisfy this requirement. Not all M-algebras satisfy this requirement.

For implication, in general M-algebras, assuming conjunction and disjunction, one could require that for any $\alpha, \beta \in M$ there be a measurement $\alpha \rightarrow \beta \in M$ such that $FP(\alpha \rightarrow \beta) = FP(\neg\alpha \vee (\alpha \wedge \beta))$, and the M-algebras defined by Hilbert spaces satisfy this requirement. Indeed, works in Quantum Logic sometimes consider more than one implication, see [**?**].

It is an important feature of M-algebras is that conjunction, disjunction and implication are defined only for commuting measurements. The next section will show that this restriction leads to a classical propositional logic. If one restricts oneself to commuting measurements, then, contrary to the unrestricted connectives of Birkhoff and von Neumann [**?**], conjunction and disjunction distribute, and, in fact, the logic obtained is classical.

6.6.2 Connectives for commuting measurements

Let us take a second look at propositional connectives in M-algebras, with particular attention to their commutation properties. We shall assume that $\langle X, M \rangle$ is an M-algebra.

Negation

Negation asserts the existence of a negation for every measurement. Let us study the commutation properties of $\neg\alpha$.

Lemma 6.13 *$\forall \alpha, \beta \in M$, if α commutes with β, then $\neg\alpha$ commutes with β.*

Proof. Assume α commutes with β. We shall see that β preserves $\neg\alpha$. Let $x \in FP(\neg\alpha)$. We have $x \in Z(\alpha)$. But $(\alpha \circ \beta)(x) = (\beta \circ \alpha)(x)$. Therefore $0 = \alpha(\beta(x))$, $\beta(x) \in Z(\alpha)$ and $\beta(x) \in FP(\neg\alpha)$. We have shown that β preserves $\neg\alpha$. By **Composition**, $(\neg\alpha) \circ \beta \in M$ and, by Lemma 6.11, $\neg\alpha$ commutes with β. ∎

Corollary 6.2 *$\forall \alpha, \beta \in M$, α and β commute iff $\neg\alpha$ and β commute iff α and $\neg\beta$ commute iff $\neg\alpha$ and $\neg\beta$ commute.*

Proof. By Lemma 6.13 and Corollary 6.1. ∎

Conjunction

We shall now define a conjunction between *commuting* measurements.

Definition 6.5 *For any* commuting *measurements $\alpha, \beta \in M$, the conjunction $\alpha \wedge \beta$ is defined by: $\alpha \wedge \beta = \alpha \circ \beta = \beta \circ \alpha$.*

By Lemma 6.11, the conjunction, as defined, is indeed a measurement.

Lemma 6.14 *For any commuting $\alpha, \beta \in M$, the conjunction $\alpha \wedge \beta$ is the unique measurement γ such that $FP(\gamma) = FP(\alpha) \cap FP(\beta)$.*

Proof. By Lemmas 6.2 and 6.7. ∎

One immediately sees that conjunction among commuting measurements is associative, commutative and that $\alpha \wedge \alpha = \alpha$ for any $\alpha \in M$.

Let us now study the commutation properties of conjunction.

Lemma 6.15 *$\forall \alpha, \beta, \gamma \in M$, that commute in pairs, $\alpha \wedge \beta$ commutes with γ.*

Proof.

$$(\alpha \wedge \beta) \circ \gamma = (\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) = \alpha \circ (\gamma \circ \beta) =$$
$$(\alpha \circ \gamma) \circ \beta = (\gamma \circ \alpha) \circ \beta = \gamma \circ (\alpha \circ \beta) = \gamma \circ (\alpha \wedge \beta)$$

∎

Disjunction

One may now define a disjunction between two commuting measurements in the usual, classical, way.

Definition 6.6 *For any* commuting *measurements $\alpha, \beta \in M$, the disjunction $\alpha \vee \beta$ is defined by: $\alpha \vee \beta = \neg(\neg\alpha \wedge \neg\beta)$.*

By Corollary 6.2, the measurements $\neg\alpha$ and $\neg\beta$ commute, therefore their conjunction is well-defined and the definition of disjunction is well-formed.

The commutation properties of disjunction are easily studied.

Lemma 6.16 *$\forall \alpha, \beta, \gamma \in M$ that commute in pairs, $\alpha \vee \beta$ commutes with γ.*

Proof. Obvious from Definition 6.6 and Lemmas 6.13 and 6.15. ∎

The following is easily proved: use Definition 6.6, **Negation** and Lemmas 6.5, 6.2 and 6.11.

Lemma 6.17 *For any commuting measurements, α and β, their disjunction $\alpha \vee \beta$ is the unique measurement γ such that $Z(\gamma) = Z(\alpha) \cap Z(\beta)$.*

Lemma 6.18 *If $\alpha, \beta \in M$* commute*, then $FP(\alpha) \cup FP(\beta) \subseteq FP(\alpha \vee \beta)$.*

The inclusion is, in general, strict.

Proof. Since $Z(\alpha \vee \beta) \subseteq Z(\alpha)$, by Lemma 6.5. ∎

Contrary to what holds in classical logic, in M-algebras we can have a state x that satisfies the disjunction $\alpha \vee \beta$ but does not satisfy any one of α or β. This is particularly interesting when α and β represent measurements of different values for the same physical quantity. In this case, one is tempted to say that such an x satisfies α not entirely but in *part* and β in some other part. In the Hilbertian formalism x is a linear combination of the two vectors $\alpha(x)$ and $\beta(x)$: $x = c_1\alpha(x) + c_2\beta(x)$. The coefficients c_1 and c_2 describe in what proportions the state x, that satisfies $\alpha \vee \beta$ satisfies α and β respectively. The consideration of structures richer than M-algebras that include this quantitative information is left for future work.

Implication

Implication ($\rightarrow$) is probably the most interesting connective. It will play a central role in our treatment of connectives.

Definition 6.7 *For any* commuting *measurements $\alpha, \beta \in M$, the implication $\alpha \rightarrow \beta$ is defined by: $\alpha \rightarrow \beta = \neg(\alpha \wedge \neg\beta)$.*

By Corollary 6.2, the measurements α and $\neg\beta$ commute, therefore their conjunction is well-defined and the definition of implication is well-formed.

The commutation properties of implication are easily studied.

Lemma 6.19 *$\forall \alpha, \beta, \gamma \in M$ that commute in pairs, $\alpha \rightarrow \beta$ commutes with γ.*

Proof. Obvious from Definition 6.7 and Lemmas 6.13 and 6.15. ∎

The following is easily proved: use Definition 6.7, **Negation** and Lemmas 6.5, 6.2 and 6.11.

Lemma 6.20 *For any commuting measurements, α and β, their implication $\alpha \rightarrow \beta$ is the unique measurement γ such that $Z(\gamma) = FP(\alpha) \cap Z(\beta)$.*

Lemma 6.20 characterises the zeros of $\alpha \rightarrow \beta$. Our next result characterizes the fixpoints of $\alpha \rightarrow \beta$ in a most telling and useful way.

Lemma 6.21 *For any commuting measurements, α and β, their implication $\alpha \rightarrow \beta$ is the unique measurement γ such that $FP(\gamma) = \{x \in X \mid \alpha(x) \in FP(\beta)\}$.*

Proof. Assume α and β commute, and $x \in X$. Now, $\alpha(x) \in FP(\beta)$ iff (by **Negation**) $\alpha(x) \in Z(\neg\beta)$ iff $(\alpha \circ (\neg\beta))(x) = 0$ iff (by Definition 6.5 and Lemma 6.13) $(\alpha \wedge (\neg\beta))(x) = 0$ iff $x \in Z(\alpha \wedge (\neg\beta))$ iff (by **Negation**) $x \in FP(\neg(\alpha \wedge (\neg\beta)))$ iff (by Definition 6.7) $x \in FP(\alpha \rightarrow \beta)$. ∎

The following is immediate.

Corollary 6.3 *For any commuting measurements α and β, if $x \in FP(\alpha)$ and $x \in FP(\alpha \rightarrow \beta)$, then $x \in FP(\beta)$.*

One may now ask whether the propositional connectives we have defined amongst commuting measurements behave classically. In particular, assuming that measurements α, β and γ commute in pairs, does the distribution law hold, i.e., is it true that $(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$. In the next section, we shall show that amongst commuting measurements propositional connectives behave classically.

6.7 Amongst commuting measurements connectives are classical

Let us, first, remark on the commutation properties described in Lemmas 6.13, 6.15, 6.16 and 6.19. Those lemmas imply that, given any set $A \subseteq M$ of measurements in an M-algebra, such that any two elements of A commute, one may consider the propositional calculus built on A (as atomic propositions). Each such proposition describes a measurement in the original M-algebra (an element of M) and all such measurements commute. We shall denote by $Prop(A)$ the propositions built on A.

We shall now show that, in any such $Prop(A)$ all classical propositional tautologies hold at every state $x \in X$.

Theorem 6.1 *Let $\langle X, M \rangle$ be an M-algebra. Let $A \subseteq M$ be a set of* commuting *measurements. If $\alpha \in Prop(A)$ is a classical propositional tautology, then $FP(\alpha) = X$.*

The converse does not hold, since it is easy to build M-algebras in which, for example, a given measurement holds at every state.

We shall use the axiomatic system for propositional calculus found on p. 31 of Mendelson's [**?**] to prove that any classical tautology α built out using only negation and implication has the property claimed. We shall then show that conjunction and disjunction may be defined in terms of negation and implication as usual. The proof will proceed in six steps: Modus Ponens, the three axiom schemes of Mendelson's system, conjunction and disjunction. The reader should notice how tightly the three axiom schemas correspond to the commutation assumption. The reader of this deaft will notice that it would of course be appropriate to use the axiom system we gave in Chapter 1. This is routine and will be done in the final version.

Lemma 6.22 *For any commuting measurements α and β, if $FP(\alpha) = X$ and $FP(\alpha \rightarrow \beta) = X$, then $FP(\beta) = X$.*

Proof. By Corollary 6.3. ▮

Lemma 6.23 *For any commuting measurements α and β,*

$$FP(\alpha \rightarrow (\beta \rightarrow \alpha)) = X.$$

Proof. Since α and β commute, for any $x \in X$: $\beta(\alpha(x)) = \alpha(\beta(x))$, therefore, by **Idempotence**, we have $\beta(\alpha(x)) \in FP(\alpha)$. By Lemma 6.21, for any x, $\alpha(x) \in FP(\beta \rightarrow \alpha)$. By the same lemma: $x \in FP(\alpha \rightarrow (\beta \rightarrow \alpha))$. ∎

Lemma 6.24 *For any pairwise commuting measurements α, β and γ*

$$FP((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))) = X.$$

Proof. By Lemma 6.21, it is enough to show that for any $x \in X$, if $y = (\alpha \rightarrow (\beta \rightarrow \gamma))(x)$, then, if we define $z = (\alpha \rightarrow \beta)(y)$, and define $w = \alpha(z)$, then we have: $\gamma(w) = w$. But since all the measurements above commute, by **Idempotence**, the state w satisfies $\alpha \rightarrow (\beta \rightarrow \gamma)$, $\alpha \rightarrow \beta$ and α. By Corollary 6.3, w satisfies β and $\beta \rightarrow \gamma$. For the same reason w satisfies γ. ∎

Lemma 6.25 *For any commuting measurements α and β,*

$$FP((\neg\beta \rightarrow \neg\alpha) \rightarrow ((\neg\beta \rightarrow \alpha) \rightarrow \beta)) = X.$$

Proof. By Lemma 6.21, it is enough to show that for any $x \in X$, if $y = (\neg\beta \rightarrow \neg\alpha)(x)$, then, if we define $z = (\neg\beta \rightarrow \alpha)(y)$ then we have: $\beta(z) = z$. But since all the measurements above commute, by **Idempotence**, the state z satisfies $\neg\beta \rightarrow \neg\alpha$ and $\neg\beta \rightarrow \alpha$. Therefore, by Lemma 6.21, $(\neg\beta)(z)$ satisfies both $\neg\alpha$ and α. Therefore $(\neg\beta)(z) = 0$ and therefore, by **Negation**, $z \in FP(\beta)$.
∎

Lemma 6.26 *For any commuting measurements α and β, $\alpha \wedge \beta = \neg(\alpha \rightarrow \neg\beta)$.*

Proof.

$$FP(\neg(\alpha \rightarrow \neg\beta)) = Z(\alpha \rightarrow \neg\beta) = FP(\alpha) \cap Z(\neg\beta) = FP(\alpha) \cap FP(\beta).$$

By **Negation**, Lemma 6.20 and **Negation**. The conclusion then follows from Lemma 6.14. ∎

Lemma 6.27 *For any commuting measurements α and β, $\alpha \vee \beta = (\neg\alpha) \rightarrow \beta$.*

Proof.

$$Z((\neg\alpha) \rightarrow \beta) = FP(\neg\alpha) \cap Z(\beta) = Z(\alpha) \cap Z(\beta).$$

By Lemma 6.20 and **Negation**. The conclusion then follows from Lemma 6.17.
∎

We have proved Theorem 6.1. The next Section will consider separable M-algebras.

6.8 Separable M-algebras

Lemma 6.28 *In a* separable *M-algebra, a measurement is classical if and only if it commutes with any measurement.*

Proof. Suppose α is classical. Consider any $x \in X$ and any $\beta \in M$. Since α is classical we know that $x \in FP(\alpha)$ or $x \in Z(\alpha)$ and $\beta(x) \in FP(\alpha)$ or $\beta(x) \in Z(\alpha)$. If $x \in FP(\alpha)$, by Lemma 6.3, $\beta(x) \in Z(\alpha)$ implies $x \in Z(\beta)$and $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$. But $\beta(x) \in FP(\alpha)$ implies $(\alpha \circ \beta)(x) = \beta(x) = (\beta \circ \alpha)(x)$.

If $x \in Z(\alpha)$ and $\beta(x) \in FP(\alpha)$, by Lemma 6.4, $\beta(x) = 0$ and $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$. If $\beta(x) \in Z(\alpha)$, then $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$.

Suppose, now that α commutes with any measurement β. By contradiction, assume $\alpha(x) \neq 0$ and $\alpha(x) \neq x$. By **Separability** there is some measurement γ such that $x \in FP(\gamma)$ and $\alpha(x) \notin FP(\gamma)$. But α and γ commute and: $(\alpha \circ \gamma)(x) = (\gamma \circ \alpha)(x) = \alpha(x)$. We see that $\alpha(x) \in FP(\gamma)$, a contradiction. ∎

Note that a measurement α is classical (see Definition 6.4) iff $Def(\alpha) = X$.

Lemma 6.29 *If α is classical, so is $\neg\alpha$. If α and β are classical, then so are $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$.*

Proof. If α is classical, $Def(\alpha) = X$ and therefore $Def(\neg\alpha) = X$. For conjunction $(\alpha \wedge \beta)(x) = (\alpha \circ \beta)(x) = (\beta \circ \alpha)(x)$. If either $\alpha(x)$ or $\beta(x)$ is 0 then $(\alpha \wedge \beta)(x) = 0$, otherwise $\alpha(x) = x = \beta(x)$ and $(\alpha \wedge \beta)(x) = x$. The definitions of disjunction and implication in terms of negation and conjunction, then ensure the claim. ∎

Chapter 7

The Local Viewpoint: States as Logical Entities

7.1 What can logic do about quantum mechanics?

The question whether we need a 'new logic' in order to reason properly in quantum theory is asked frequently. Do we have to depart from classical logic in building 'quantum logic' and if so, how? The answer to this question given by most physicists is that we do not. In fact, physicists put quantum mechanics to good use in an unprecedently successful way, and in this do they not use classical logic? Popper in [51] denies any need to depart from classical logic in order to reason properly in quantum mechanics.

Why then did the question arise at all? As we already saw, the question of 'the logic of quantum mechanics' was, in the scientific literature, first raised by Birkhoff and von Neumann in their seminal 1936 paper. As discussed at length in Chapter 5, their motivation for trying to discover the 'logic of quantum mechanics' was the fact that they considered the novel features of quantum mechanics such as the uncertainty relations to be logical in nature. Since these features are not reflected in classical logic there is, according to Birkhoff-von Neumann, a need to construct a (logical) 'calculus' in which they are actually represented.

Later on it was Putnam, Finkelstein and others who put forward a view of quantum logic which for some time attracted considerable attention. Central to this paradigm is the idea that logic may be "empiricial". Putnam and his followers argued that the role of logic in quantum mechanics was similar to that of geometry in the theory of relativity. In the theory of relativity Euclidean geometry, which in Newtonian physics was still considered a priori, had to be revised on empirical grounds. In quantum mechanics, Putnam argued, it is (classical) logic that needs revision on empirical grounds. Similar to the way

the theory of relativity teaches us the "real" geometry it is quantum mechanics that teaches us the "real" logic. This is undoubtedly an attractive idea which, however we do not pursue in this book, at least not in the form it was put forward by Putnam and his followers. We will, however, come back to this in the last chapter.

In this book we adopt a different attitude which seems already implicit in the Birkhoff-von Neumann paper. Let us recall what they write in the Introduction: "The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic".

In fact, this is the task we pose ourselves: searching for logical structures in quantum mechanics. The procedure is this. We take a close look at Hilbert space and, as a result, identify and study certain logical structures implicit in Hilbert space. We then pursue the question whether these logical structures represent essential features of the formalism of quantum mechanics. This is to serve a twofold purpose. First, as already pointed out, we then have a precise framework at our disposal in which the somewhat vague intuitions that are suggested by Chapter 8 can be reflected and stated in a precise way. This logical framework can then play the role of a platform of discourse on which we can give a precise logical meaning to certain intuitions that arise naturally in view of the 'strange' aspects of quantum reality described in Chapter 8.

Can these structures shed light on the formalism of quantum mechanics? We would like to make the point that they in fact can. So the answer to the question asked in the title of this section is that logic (as a science) can detect and study logical structures in the formalism of quantum mechanics which are essential for the understanding of the formalism itself.

Are there any guidelines that may help us in our search for these structures? Are there any traits of quantum mechanics itself that could suggest certain directions of investigation. Let us speculate a bit about this.

As a good starting point we may look at the relationship between classical and quantum mechanics. We may start by analysing the way how quantum mechnics departs from classical mechanics. Given that quantum mechanics, as is often claimed more or less vaguely, does not conform to classical logic, then it is reasonable to ask how the transition from classical to quantum mechanics is reflected in the logical structures we are looking for.

There are various ways of viewing the relationship between classical and quantum mechanics. Since in classical mechanics we have no uncertainty relations it is the uncertainty relations that are often regarded as constituting the essential difference. Another crucial difference concerns the role of measurement. In classical mechanics a measurement does not involve a change of the state of the system measured. The fact that in quantum mechanics measurement does, in general, involve such a change of state is undoubtedly an essential difference between classical and quantum physics. Classical mechanics is often considered to be a limiting case of quantum mechanics as classical (Newtonian) mechanics is a limiting case of the theory of relativity. We may ask the question how these observations are reflected in the logical structures we find. What

are uncertainty relations from the logical point of view?. We already remarked in chapter 2 that, logically, the presence of uncertainty relations is reflected as non-monotonicity of the logical structures implicit in the formalism of quantum mechanics. In Chapter 6 we took the fact that there may be a change of state in quantum measurement as an inspiration for the dynamic view of propositions as acting on states rather than just being true or false in these states.

There is, however, a general feeling expressed in a vast body of literature, popular scientific and seriously philosophical alike, that the fairly obvious differences between classical and quantum mechanics mentioned above are not the whole story. Rather the general impression seems to be that the way how quantum mechanics departs from classical mechanics touches on deeper ground. In the next chapter we will discuss the famous Einstein-Podolsky-Rosen (EPR) argument put forward in a paper entitled "Can the quantum-mechanical description of reality be considered complete?" see [13]. In the EPR argument the term "element of reality" plays a crucial role. EPR take it for granted that (physical) reality is to be viewed as consisting of separate "elements of reality". And, in fact, once this fragmented view of reality is accepted it is hard to avoid the EPR conclusion that quantum mechanics does not provide a complete description of physical reality. Therefore, if quantum mechanics is in fact a complete description of physical reality as seems to be generally assumed nowadays, then something must be wrong with this view of reality. It seems that the way quantum mechanics departs from classical mechanics is of an even more profound nature than the way the (special) theory of relativity departs from lassical mechanics. In the latter case we 'just' have to abandon our views on space and time. In the case of quantum mechanics it seems that we have to abandon our views on the very nature of reality. This is all pervading the literature on the foundations of quantum mechanics be it popular scientific or seriously philosophical. It is the intuition of oneness, interconnectedness and wholeness, which is prevalent in Eastern thought for instance, that finds strong support in quantum mechanics. But this is the realm of intuition and metaphor, perhaps even of philosophy, and it is hard to make something scientific of this at the level of ordinary discourse.

How can we reflect the shift in our perception of reality which is forced upon us in the transition from classical mechanics to quantum mechanics at the level of logic? A possible answer is this. Classical mechanics and classical logic conform to each other and the view of reality that underlies classical mechanics also underlies classical logic. If, as seems to be the case, our 'classical' view of reality is to be revised in quantum mechanics, we must ask the question whether logic, i.e. the quantum logic to be constructed, can account for quantum mechanics if it does not reflect this shift. In this book we propose a way of departing from classical logic for the sake of constructing quantum logic which may be regarded as reflecting this intuition. This is one important feature of our way of departing from classical logic.

7.2 States as logical entities

The concept of a state of a physical system plays a fundamental role both in classical and in quantum physics. In our study of M-algebras, which constitute our first abstraction from the Hilbertian formalism, we treated the concept of a state of a physical system as a primitive notion. In this chapter we ask the question what *is* a state from the logical point of view. Can we view the states of a physical system as logical entities themselves and if so, what is the nature of these logical entities?

In classical physics, the concept of a state is, from the logical point of view, unproblematic. Logically, in classical physics a state is a complete classical theory. It can be identified with the set of all physical statements *true* about the system. In this sense the state of the system at a certain point in time fully contains all the information about the system. What in classical mechanics is particularly convenient is the fact that once we know the momenta and the positions of the particles constituting the system, we know all relevant physical properties. Therefore, from the logical point of view, the state can be described by a single proposition, namely by the proposition specifying all values of the momenta and positions at a given time. From this we can then compute (deduce) the values of all relevant physical quantities. This is what in classical mechanics is known as *phase space*. So, the logical analogue of the concept of a state in classical mechanics is that of a *complete classical theory*.

This simple concept of state is based on the view underlying classical mechanics that a physical system *possesses* certain properties and does *not possess* others. The propositions expressing the physical properties a physical system can possess accordung to classical mechanics may be viewed as having the form $A = \mu$, where A denotes a physical quantity (observable) such as position, momentum, energy... In any given state, for any observable A the propsition $A = \mu$ is true for exactly one (real) value μ. For any value $\rho \neq \mu$ the proposition $A = \rho$ is false which is equivalent to $\neg A = \rho$.

For any physically meaningful property the system either possesses it or not. Any given proposition holds or does not hold at any given point in time. In the latter case it is, by classical logic, the negation of the proposition that holds. Take for instance position Q and consider the propsition $Q = \mu$, take moreover momentum P and consider the proposition $P = \lambda$. Then according to classical mechanics these propositions or their negations are true. We may for instance have $Q = \mu$ is true and $P = \lambda$ true. In this case we can deduce any other proposition true about the system. We may for instance infer a proposition of the form $\neg(E = \rho)$, where E denotes kinetic energy. Obviously, this concept of a state rests on classical logic and in particular on the notion of truth underlying classical logic.

How can we, in classical physics, know if the system possesses a certain property, how can we know that, say $A = \mu$ is true? The answer is that given this proposition we can at least in principle fnd out whether it is true via measurement. $A = \mu$ is true if and only if a measurement of the physical quantity A yields the value μ. Whenever we measure A we always get μ and no

other value as a result of measurement.

Now, what's different in quantum mechanics? Why can't we represent the state of a quantum system analogously, namely by the set of those propositions that are true or false in this state? The reason is that, in quantum mechanics, the term 'is true' is far less clear. In classical mechanics we said that 'to be true' may, roughly, be taken as 'be measured' or at least 'to be measurable'. In quantum mechanics things aren't that simple. Given a state x in quantum mechanics and a proposition $A = \mu$. Suppose we perform a measurement of an observable A in x. Then the following three cases may occur. First, the probability to measure μ is 1 in which case we may reasonably say that $A = \mu$ is true (in state x). Second, the probability to get μ as a result of measurement may be 0. In this case we may reasonably say that $A = \mu$ is false or, equivalently, $\neg(A = \mu)$ is true. In quantum mechanics there is, however, a third case which marks the difference with classical mechanics. Namely, the probability to get μ may be greater than zero and smaller than one. Let us for the moment call these propositions contingent with respect to x. It is then obviously insufficient to represent the state x by the set of those propositions of the form $A = \lambda$ that are true or false in x because this does not give us any infomation about the contingent propositions and their probabilities. It seems that a proper representation of a quantum state must specify probabilities. In a purely logical treatment of the concept of a physical state we should, however, try to avoid specifying probabilities.

Let us now reflect on the problem of representing states within the framework of M-algebras. Given an M-algbra $\langle X, M\rangle$ and a state $x \in X$. Again, it is insufficient to represent x as the set of those propositions α such that $x \in FP(\alpha)$ or $x \in FP(\neg\alpha$. In fact, there exist, generally, propositions such that neither $x \in FP(\alpha)$ nor $x \in FP(\neg\alpha)$. These propositions act on x in that they neither leave it unchanged nor send it to zero. They are neither 'true' nor 'false'.

We may think of the action of propositions on states in an M-algebra as a sort of *coming true* rather than *being true*. We may say that α *is* true in x if $x \in FP(\alpha)$ and x is false in x if $x \in FP(\neg\alpha$. Otherwise, i.e. in case that $\alpha(x) = y \neq x$ and $\alpha(x) \neq 0$ we may say that α *comes* true in x. Thus α *comes* true in x if it *is* true in $\alpha(x)$. Hence the representation of the state x must give us information not just on what is true or false in x but about what comes true in x. Thus in an M-algebra it is the coming true of a proposition that replaces or generalises the being true of a proposition in classical logic. This is, in an M-algebra, the dynamic analogue of the static concept of being true in classical logic. However, coming true in x involves a different state which in turn must be specified. Thus, intuitively, we must require the logical entity representing a state x as also specifying other states, namely all those states in which a proposition may end up true when it *comes* true in state x.

Technically speaking, it's as follows. The logical entity representing a state in classical logic, namely a complete theory, contains (encodes) all propositions true or false in this state. When, however, we are concerned with propositions that act on the state or, as we said, have the property of coming true rather than being true in state x, then the logical representation of x must encode

all propositions that come true at x, in other words the logical representation of a state x must encode the action of the propositions on x. The action of a proposition on x however yields a new state y and therefore the state x must encode other states. So we inevitably hit here on the phenomenon of *encodedness of states in other states* which will play a dominant role in our study of *holistic logics* introduced in chapter 9. We will see that, there, a state is a logical entity that encodes (almost) all other states including itself.

7.3 M-algebras and their languages

In this chapter we will study several special types of M-algebras. We will define these M-algebras by introducing additional connectives between arbitrary measurements. Generally, we have only one connective in an M-algebra that is defined for arbitrary measurements, namely negation. In this chapter we will consider M-algebras which for instance have an implication defined between arbitrary measurements. We call these M-algebras Implication M-algebras. We will also consider M-algebras having a generally defined conjunction. We call these algebras Conjunction M-algebras.

Let us now desccribe the proper (logical) languages suited to these M-algebras. Given an M-algebra $\mathcal{A} = \langle X, M \rangle$. Then consider a set Var_X of variables of the same cardinality as X. Suppose we have an M-algebra with additional connectives, say conjunction, i.e. a Conjunction M-algebra. The proper language of the M-algebra $\mathcal{A}$ denoted by $\mathcal{L}^{\neg,\wedge}$ is then defined by the following clauses.

- Every variable, i.e. an element of Var_X is a formula of $\mathcal{L}^{\neg,\wedge}$.
- If α is a formula of $\mathcal{L}^{\neg,\wedge}, sois\neg\alpha$
- If α and β are formulas of $\mathcal{L}^{\neg,\wedge}$, so is $\alpha \wedge \beta$.

It is now obvious how to define the corresponding language $\mathcal{L}^{\neg,\leadsto} ofanM - algebraorthelanguage$L$^{\neg,\leadsto,\wedge}$ of an M-algebra which is both an Implication M-algebra and a Conjunction M-algebra.

We can describe this somewhat informally as follows. A precise definition is routine and will be given in the final version. Given an M-algebra $\mathcal{A} = \langle X, M \rangle$. Assume we have certain connectives in $\mathcal{A}$, say $\neg$, $\wedge$, an implication $\leadsto$... Note hat in an M-algebra connectives are algebraic operations, i.e. functions from $M \times M$ to M. We need to construct a propositional lnguage suited to $\mathcal{A}$. We can do this as follows. Choose a set of propositional variables Var of the same cardinality as M and let $V : Var \to M$ be a bijection. Consider the propositional language $\mathcal{L}_{\mathcal{M}}$ built up from Var and the usual connectives as described in chapter 2. The function V can then be extended to $\mathcal{L}_M$ in unique way such that $V(\neg\alpha)) = \neg V(\alpha)$, $V(\alpha \wedge \beta) = V(\alpha \wedge \beta)$...Note that on the left hand side the connectives are logical connectives in the usual sense, namely the connectives of the propositional language $\mathcal{L}_M$ whereas the connectives on the

right hand side are the corresponding algebraic operations of the M-algebra. Since V is a bijection respecting the logical connectives we may identify α, i.e. a formula of $\mathcal{L}_M$ with $V(\alpha)$ and denote $V(\alpha)$ again by α.

7.4 Implication M-algebras

In our study of M-algebras we dealt for the most part with what we call *global properties*. This also applies to the Birkhoff-von Neumann paper. What is a global property? A global property is a property concerned with sets of states. We for instance proved in Chapter 6 that in any M-algebra commuting propositions obey classical logic. Since propositions are functions from the set of states into itself this property of an M-algebra is a global property.

In this section we focus on what we call *local properties*. This is what we mean by *local viewpoint*. By a local property we mean a property concerned with a certain fixed state, properties of individual states so to speak rather than properties concerning sets of states. Some of the properties we already dealt with in connection with M-algebras *are* local properties. Recall for instance the axiom of Interference. The study of local properties of M-algebras gives us insight into the logical nature of physical states.

However, the concept of an M-algebra as introduced and studied in chapter 6 is still too general for an investigation of this sort. For this we need to consider more special structures. We will call the structures suitable for such an investigation Implication M-algebras.

In our study of M-algbras in Chapter 6 we admitted only one generally defined connective, namely negation, i.e. for any proposition α its negation $\neg\alpha$ is defined. All the other connectives, conjunction, disjunction, implication are defined for commuting propositions only. And, with this restriction, we proved that they behave classically. This generalises the well known fact that a set of mutually commuting projections in Hilbert space forms a Boolean algebra.

In an Implication M-algebras we allow for another connective defined between arbitrary propositions, namely implication, which we will denote by $\rightsquigarrow$. This connective has, in terms of measurement, the following intuitive meaning: $\alpha \rightsquigarrow \beta$ means: "If we measure α we (also) get β". We will see that for commuting measurements this implication will coincide with the implication we already have in this case.

In Chapter 6 we proved the following.

Let $\langle X, M\rangle$ be an M -algebra, $x \in X$ and $\alpha, \beta \in M$ be two commuting measurements. Then we have

$$x \in FP(\alpha \to \beta) \text{ iff } \alpha(x) \in FP(\beta)$$

Given an M-algebra $\langle X, M\rangle$ and consider a fixed state $x \in X$. We now define a binary relation $\mid\!\sim_x$ between arbitrary measurements (propositions) as follows.

Definition 7.1 *Let $\langle X, M\rangle$ be an M-algebra and $x \in X$. Then define the binary relation $\mid\!\sim_x$ as follows. Given α, $\beta \in M$. Then we say $\alpha \mid\!\sim_x \beta$ if $\alpha(x) \in FP(\beta)$.*

We may, intuitively, think of these relations as consequence relations although in the general situation of M-algebras we cannot expect them to satisfy the minimal conditions of Chapter 2. But we will see in chapter 10 that in the case of a Hilbert space these conditions are in fact satisfied and thus these binary relations may rightfully called consequence relations.

In the sequel we write $\mid\!\sim_x \alpha$ for $\top \mid\!\sim_x \alpha$.

Lemma 7.1 *$\alpha \mid\!\sim_x \beta$ iff $\mid\!\sim_{\alpha(x)} \beta$ iff $\alpha(x) \in FP(\beta)$*

Proof. By definition. ■

Let us now say what we mean by an Implication M-algebra.

Definition 7.2 *We call an M-algebra $\langle X, M\rangle$ an Implication M-algebra if there exists a function $I : M \times M \to M$ such that for any $x \in X$, $\alpha \mid\!\sim_x \beta$ iff $x \in FP(I(\alpha, \beta))$ ir equivalently $\alpha(x) \in FP(\beta)$.*

Note that the function I is, if it exists, uniquely determined because a measurement is uniquely determined by its set of fixpoints. Call it the implication of the M-algebra. As aleady mentioned we also write $\alpha \rightsquigarrow \beta$ for $I(\alpha, \beta)$.

Recall that orthomodular spaces and in particular Hilbert spaces give rise to an M-algebra in a natural way. By Proposition 3.4 the implication I is given by $I(\alpha, \beta = \alpha^{\perp} \vee (\alpha \wedge \beta)$, i.e. the Sasaki hook. We will see shortly that this is not accidental.

Lemma 7.2 *If α and β commute we have $\alpha \rightsquigarrow \beta = \alpha \to \beta$.*

Proof. By the definition of $\rightsquigarrow$ and Lemma ?. ■

7.5 Conjunction M-algebras

Definition 7.3 *We call an M-algebra $\langle X, M\rangle$ a Conjunction M-algebra if there exists a function $C : M \times M \to M$ such that for any $\alpha, \beta \in M$ we have $FP(C(\alpha, \beta)) = FP(\alpha) \cap FP(\beta)$. We call such a function a conjunction. For $C(\alpha, \beta)$ we also write $\alpha \wedge \beta$.*

Again, if an M-algebra admits a conjunction, this conjunction is uniquely determined because measurements are uniquely determined by their set of fixpoints.

The following lemma follows from the definition of a conjunction and lemma 6.14.

Lemma 7.3 *Let $\mathcal{A}$ be an M-algebra and C a conjunction of $\mathcal{A}$. Then for commuting measurements C agrees with the conjunction (already) defined.*

Theorem 7.1 *Given a conjunction M-algebra $\mathcal{A} = \langle X, M\rangle$ with conjunction C. Then $\mathcal{A}$ admits a (unique) implication I. Namely we have $I(\alpha, \beta) = \neg\alpha \vee (\alpha \wedge \beta) = \alpha \to (\alpha \wedge \beta)$.*

Note that for any $\alpha, \beta \in M$ $\neg\alpha \vee (\alpha \wedge \beta)$ is defined. Namely we have $FP(\alpha) \subset FP(\alpha \wedge \beta)$. Hence α and $\alpha \wedge \beta$ commute by Lemma 6.12 and thus by Lemma 6.13 $\neg\alpha$ and $\alpha \wedge \beta$ commute. The theorem says that any conjunction M-algebra is also an Implication M-algebra.

Proof. It suffices to prove that $FP(\alpha \to \beta) = \{x \mid \alpha(x) \in FP(\beta)\}$. By Lemma 6.21 we have $FP(\alpha \to \beta) = \{x \mid \alpha(x) \in FP(\alpha \wedge \beta)\}$. By the definition of conjunction this set is equal to $\{x \mid \alpha(x) \in FP\alpha) \cap FP(\beta)\}$. Since $\alpha(x) \in FP(\alpha)$, it follows that $FP(\alpha \to \beta) = \{x \mid \alpha(x) \in FP(\beta)$. ■

Recall from chapter 3 that the connective $\rightsquigarrow$ defined by $\alpha \rightsquigarrow \beta = \neg\alpha \vee (\alpha \wedge \beta$ is called the *Sasaki hook.*

The proof of the following lemma is an easy exercise in classical propositional logic.

Lemma 7.4 *The Sasaki hook is classically equivalent to material implication, i.e,* $\vdash (\alpha \rightsquigarrow \beta) \leftrightarrow (\alpha \to \beta)$

7.6 Strongly separable M-algebras

Definition 7.4 *Let* $\langle X, M\rangle$ *be an M-algebra and* $x \neq 0, x \in X$. *We say that a measurement* e *is a pointer to* x *if* $FP(e) = \{x, 0\}$

Again, note that given x then a pointer to x is uniquely determined. We denote it by e_x.

Let $\langle X, M\rangle$ be an M-algebra, let e_x be a pointer to x. Then for any $x, y \in X$ we have $e_x(y) = 0$ for $x \in Z(e_x)$, otherwise $e_x(y) =$ x.

Proof. The first claim expresses a familiar property of negation. For the second claim suppose that not $y \in Z(e_x)$. Then $e_x(y) \neq 0$ and $e_x(y) \in FP(e_x). But, since \mathrm{FP(e_x)} = \{x, 0\}$, it follows that $e_x(y) = x$. ■

Definition 7.5 *We call an M-algebra* $\langle X, M\rangle$ *a srongly separable M-algebra if every* $x \in X$ *has a pointer.*

Proposition 7.1 *A strongly separable M-algebra* $\mathcal{A} = \langle X, M\rangle$ *is separable.*

Proof. Given two ditinct $x, y \in X$. Then $x \in FP(e_x)$ and not $\in FP(e_y)$ and vice versa. It follows that $\mathcal{A}$ is separable. ■

Lemma 7.5 *Given a strongly separable M-algebra* $\langle X, M\rangle$. *Given* $x, y \in M$. *Then* $e_x(y) = 0$ *implies* $e_y(x) = 0$

Definition 7.6 *Let* $\langle X, M\rangle$ *be an strongly separable M-algebra. Let* $x, y \in X$. *Then wee say that* x *and* y *are orthogonal if* $e_x(y) = 0$.

By the above proposition the relation of orthogonality is symmetric.

7.6.1 States encode each other

Proposition 7.2 *Let $\langle X, M\rangle$ be a strongly separable Implication M-algebra and let x, y be non-orthogonal states. Then $\mid\!\sim_x \alpha$ implies $\mid\!\sim_y e_x \rightsquigarrow \alpha$, equivalently $e_x \mid\!\sim_y \alpha$*

Proof. Suppose $\mid\!\sim_x \alpha$ and let $y \in X$. We know that $\sigma_x(y) = x$ or $\sigma_x(y) = 0$. In the first case we have $\mid\!\sim_{\sigma_x(y)} \alpha$. Thus $\sigma_x \mid\!\sim_y \alpha$. This is equivalent to $\mid\!\sim_y \sigma_x \rightsquigarrow \alpha$ In the second case we have $\mid\!\sim_{\sigma_x} \alpha$ since $\sigma_x = 0$ and thus $\sigma_x \mid\!\sim_x \alpha$. ■

Let us at this point recall our intuitive reflections on the logical nature of (physical) states. We there arrived at the conclusion that, logically, (quantum) states must be represented in such a way that they encode other states. The above may be viewed as one way of formally expressing this property. We will elaborate on this phenomenon of mutual encodedness in Chapter 9 (on holistic logics.

7.6.2 Positive and Negative Introspection

Lemma 7.6 *Given an M-algebra $\langle X, M\rangle$ and $x \in X$ with pointer σ_x. Then the measurements e_x, $\neg e_x$, $\top$, $\bot$ mutually commute. They form a Bolean algebra in a natural way.*

Proof. is fairly obvious BUT DO IT ■

Lemma 7.7 *Let $\langle X, M\rangle$ be a strongly separable Implication M-algebra and $x \in X$. Then we have for any measurement α*

- $\mid\!\sim_x \alpha$ *iff* $e_x \rightsquigarrow \alpha = \top$ *and thus* $\neg(e_x \rightsquigarrow \alpha = 0$
- $\not\mid\!\sim_x \alpha$ *iff* $e_x \rightsquigarrow \alpha = \neg e_x$ *and thus* $\neg(e_x \rightsquigarrow \alpha) = e_x$

Proof. For (i) suppose $\mid\!\sim \alpha$. Then we have by the above lemma that $FP(e_x \rightsquigarrow \alpha) = X$ and thus $e_x \rightsquigarrow \alpha = \top$.

For (ii) assume $\not\mid\!\sim_x \alpha$ and let y be orthogonal to x. Then we have by the argument used above that $\mid\!\sim_y e_x \rightsquigarrow \alpha$. Now suppose that y is not orthogonal to x. Then we have $e_x(y) = x$. Assume $\mid\!\sim_y e_x \rightsquigarrow \alpha$. This implies $\mid\!\sim_x \alpha$ contrary to the assumption. Thus $FP(e_x \rightsquigarrow \alpha = FP(\neg e_x)$. But this means $e_x \rightsquigarrow \alpha = \neg e_x$.

This proves the lemma. ■

Corollary 7.1

- $\mid\!\sim_x \alpha$ *iff* $\mid\!\sim_x e_x \rightsquigarrow \alpha$
- $\not\mid\!\sim \alpha$ *iff* $e_x \neg(e_x \rightsquigarrow \alpha)$

We may view the (object formulas) $e_x \rightsquigarrow \alpha$ and $\neg(e_x \rightsquigarrow \alpha)$ as expressing metastatements. Namely, we may, if we think of $\mid\!\sim_x$ as a consequence relation, view $e_x \rightsquigarrow \alpha$ as saying 'α is provable in $\mid\!\sim_x$' and '$\neg(e_x \rightsquigarrow \alpha)$ as saying 'α is not provable in $\mid\!\sim_x$.

7.6.3 States as self-contained logical entities

Given a strongly separable Implication M-algebra $\langle X, M\rangle$. Given a fixed $x \in X$. Let us thunk of the relation $\mid\!\sim_x$ as a consequence relation, although, as already mentioned, these relations do not necessarily satisfy the minimal cconditions stated in Chapter 2. So, intuitively, we regard, logically, a stae x as a sort of state of provability. We then define a meta-language ML_x in which we can make statements about the state x, more precisely about provability in state x. Let us give some motivation for the way we define this language. What do we expect it to be capable of expressing?

Given two propositions (measurements) α and β such that $\alpha \mid\!\sim_x \beta$. This says in ordinary language: "β follows from α" or synonymously "β is derivable from α". We want the metalanguage, which is to be a formal language, to be able to express this. We therefore need a meta-connective which we denote by DER reminiscent of "derivable" expressing this. Thus in the metalanguage $DER(\alpha, \beta)$ says that $\alpha \mid\!\sim_x \beta$. Clearly, we want to combine statements of the form $DER(\alpha, \beta)$ by the usual propsitional connectives so that we may able to express say "If β is derivable from α, then γ is not derivable from β."

Another feature of the metalanguage is that it should not contain statements of the object language, i.e. measurements. The meta-language is intended to be *about measurements*. A measurement (proposition) itself is thus not a metastatement. The metalanguage will have to be constructed in such a way that measurements, i.e. object formulas, are in the scope of the metaoperator DER. In the sequel we will identify measurements with formulas of a propositional language which we call the object language.

Intuitively, $DER_x(\alpha, \beta)$ means 'β is derivable from α in $\mid\!\sim_x$'.

Definition 7.7
- *(i) If α, β are wffs of the object language, then $DER_x(\alpha, \beta) \in ML_x$.*
- *If α is a wff of the object language and $\varphi \in ML_x$, then $DER_x(\alpha, \varphi) \in ML_x$ and $DER_x(\varphi, \alpha) \in ML_x$.*
- *If $\varphi, \psi \in ML_x$, then $DER_x(\varphi, \psi) \in ML_x$.*
- *If $\varphi, \psi \in ML_x$, so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$.*

We use the following abbreviations:

$$PROV_x\alpha =: DER_x(\top, \alpha)$$

$$CON_x\alpha =: \neg PROV_x\neg\alpha$$

$$EQUIV_x(\alpha, \beta) =: DER_x(\alpha, \beta) \wedge DER_x(\beta, \alpha)$$

We now define a natural translation of the meta-language ML_x into the object language.

Definition 7.8 *We define for a given state x the translation $'$ as follows.*

- *(i) If* $\varphi = DER_x(\alpha, \beta)$ *where* α *and* β *are wff of the object language,* $\varphi' = e_x \rightsquigarrow (\alpha \rightsquigarrow \beta)$
- *(ii) If* $\varphi = DER_x(\alpha, \psi)$*, where* α *is a wff of the object language and* $\psi \in ML$*, then* $\varphi' = e_x \rightsquigarrow (\alpha \rightsquigarrow \psi')$*; analogously for the case* $DER_x(\psi, \alpha)$
- *(iii) If* $\varphi = DER_x(\psi, \rho)$*, where* $\psi, \rho \in ML$ $\varphi' = e_x \rightsquigarrow (\psi' \rightsquigarrow \rho')$
- *(iv) If* $\varphi = \neg\psi$*,* $\varphi' = \neg(e_x \rightsquigarrow \psi')$*,*
- *(v) If* $\varphi = \psi \wedge \rho$*,* $\varphi' = \psi' \wedge \rho'$*; analogously for the other connectives.*

We now define the the notion of truth for ML_x in a natural way. This definition of truth is in the spirit of what Smullyan calls a self-referential interpretation in [59] and

We study truth as a local notion. So we assume a fixed state x.

Definition 7.9
- *(i) If* $\varphi = DER_x(\alpha, \beta)$*, where* α, β *are wffs of the object language, then TRUE* φ *iff* $\alpha \mid\!\sim_x \beta$
- *(ii) If* $\varphi = DER_x(\alpha, \psi)$*, where* α *is a wff of the object language, then TRUE* φ *iff* $\alpha \mid\!\sim_x \psi'$*; analogously for the case* $DER_x(\psi, \alpha)$
- *(iii) If* $\varphi = DER_x(\psi, \rho)$ *for* $\psi, \rho \in ML$*, then TRUE* φ *iff* $\psi' \mid\!\sim_x \rho'$*.*
- *(iv) If* $\varphi = \neg\psi$*, then TRUE* φ *iff not TRUE* ψ*; analogously for the other connectives.*
- *TRUE* $\varphi \wedge \psi$ *if TRUE* φ *and TRUE* ψ
- *TRUE* $\varphi \vee \psi$ *if TRUE* φ *or TRUE* ψ
- *TRUE* $\varphi \rightarrow \psi$ *if not TRUE* φ *or TRUE* ψ

Theorem 7.2 *Given a strongly separable Implication M-algebra* $\langle X, M\rangle$*,* $x \in X$*. Then we have for any* φML_x*,* φ *is TRUE iff* $\mid\!\sim_x \varphi'$

In the follwing proof we omit the subscript x wherever it should occur.

Proof. By induction on the construction of the formulas of ML.
(i) Case $\varphi = DER(\alpha, \beta)$, where and β are wff of the object language. By definition $TRUE$ φ means $\alpha \mid\!\sim \beta$. But this says $\mid\!\sim \alpha \rightsquigarrow \beta$, which is equivalent to $\mid\!\sim \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$. But this says that $\mid\!\sim \varphi'$.
(ii) Case $\varphi = DER(\alpha, \psi)$. Suppose $TRUE$ φ. By definition this says $\alpha \mid\!\sim \psi'$ or equivalently $\mid\!\sim \sigma \rightsquigarrow (\alpha \rightsquigarrow \psi')$. But this is exactly what $\mid\!\sim \varphi'$ means.
(iii) Case $\varphi = DER(\psi, \rho)$. The proof is analogous to (ii).
(iv) Case $\varphi = \neg\psi$. This is the crucial case. $TRUE$ φ means that not $TRUE\psi$. By the induction hypothesis this is equivalent to not $\mid\!\sim \psi'$, which by 'provability of unprovability' says that $\mid\!\sim \neg(\sigma \rightsquigarrow \psi')$. But this means $\mid\!\sim \varphi'$.
(v) Case $\varphi = \psi \vee \rho$. First note that $\varphi' = \psi' \vee \rho'$. Suppose $TRUE$ φ. It follows

that $TRUE\ \psi$ or $TRUE\ \rho$. Without loss of generality assume $TRUE\ \psi$. By the induction hypothesis we have $\mid\!\sim \psi'$ and thus $\mid\!\sim \psi' \vee \rho'$. But this says $\mid\!\sim \varphi'$.

For the other direction suppose $\mid\!\sim \varphi'$. We need to prove that $TRUE\ \varphi$. There is a problem here, namely that, generally, $\mid\!\sim \psi' \vee \rho'$ does not imply that $\mid\!\sim \psi'$ or $\mid\!\sim \rho'$. To overcome this obstacle we first observe by inspecting the definition of the translation that any formula occurring as a translation is of the form $\sigma \rightsquigarrow$ ... or $\neg(\sigma \rightsquigarrow$ ... or a Boolean combination of such formulas. It then follows by Lemma 3 that the formulas (measurements) $[\psi']$ and $[\rho']$ are of the form $\top$, $\bot$, σ, $\neg\sigma$. We can thus treat this case by checking all combinations.

Suppose for instance that $\psi' = \top$ and $\rho' = \neg\sigma$. Then $\mid\!\sim \psi' \vee \rho'$ says $\mid\!\sim \top \vee \neg\sigma$, which is equivalent to $\mid\!\sim \top$, i.e. $\mid\!\sim \psi'$. It follows by the induction hypothesis that $TRUE\ \psi$ and thus $TRUE\ \varphi$.

The other combinations are checked analogously. ∎

7.6.4 Conjunction: the source of classical inconsistency in M-algebras

We already mentioned that in [33] Kochen and Specker made, as a byproduct of their work on the problem of hidden variables in Quntum Mechanics, an observation that sheds an interesting light on the relationship between classical logic and Birkhoff-von Neumann quantum logic. They present a classical tautology φ in 117 propositional variables which under a certain valuation of its variables as subspaces of three-dimensional Hilbert space represents the zero space.

In this section we study this phenomenon of classical inconsistency in the general setting of M-algebras and prove that the phenomenon discoverd by Kochen and Specker is not an accident. We prove a general theorem from which, in the case of a finite-dimensional orthomodular space and thus in the case of a finite -dimensional Hilbert space, we get the existence of a quantum tautology which is a classical contradiction. LOOSELY SPEAKING, WE WILL SEE THAT CONJUNCTION IS THE CULPRIT. However, a word of caution is in order here. What the corollary to our theorem says is that for any finite-dimensional Hilbert space there exists a classical tautology representing the zero space. And in fact such a formula can be readily presented explicitly. What we do not get, however, is that this tautology has the additional property that the connectives $\wedge$ and $\vee$ combine commuting measurements only as is the case in the Kochen-Specker tautology as well as Schütte's tautology.

Does the phenomenon described above really come as a surprise? In view of the follwing intuitive consideration it does not. Recall Birkhoff and von Neumann's view of the connectives of the logic of experimental propositions. Given two experimental propositions α and β mathematically represented by the closed subspaces a and b respectively. Suppose the observation spaces for α and β are given by the observables A and B respectively. So α says something like this: "The observable A has a value in some some subset S_a of its observation space" and β says: "The observable B has a value in some subset S_b of its observation space." Now, according to Birkhoff and von Neumann, the closed subspace $a \cap b$ again represents an experimental proposition, say γ. The

point now is this. Assume α and β are compatible propositions then if γ and α and β have a common observation space coming from a certain observable C. $\alpha \wedge \beta$ says something like: "The observable C has a value in $S_a \cap S_b$. The 'topic' of the experimental propositions α and β remains the same so to speak, namely the observable C is the topic of both propositions. If, however, α and β do not commute, i.e. are not compatible, then the observation space of the experimental proposition γ may come from an observable D unrelated to the observables A and B. It is thus not surprising that this 'change of topic' involved in the conjunction of non-commuting propositions can cause the extreme deviation from classical logic described above.

In view of the fact that any Conjunction M-algebra admits an implication our goal is the following theorem.

Lemma 7.8 *Let $\mathcal{A} = \langle X, M \rangle$ be a strongly separable Conjunction M-algebra. Given any $x \in X$ with pointer e_x. Suppose that $\Sigma_g \cup \{e_x\}$ is consistent. Then we have for any α*

$$\Sigma_g \cup \{e_x\} \vdash \alpha \textit{ iff } \mid\!\sim_x \alpha$$

Proof. First note that a conjunction M-algebra admits a unique implication $\rightsquigarrow$, namely $\alpha \rightsquigarrow \beta = \neg\alpha \vee (\alpha \wedge \beta)$. Further recall that $\alpha \mid\!\sim_x \beta$ iff $\mid\!\sim_x \alpha \rightsquigarrow \beta$.

Assume $\mid\!\sim_x \alpha$. Then we have $\sigma_x \rightsquigarrow \alpha \in \Sigma_g$ by ?. It follows that $\Sigma_g \vdash e_x \rightsquigarrow \alpha$. Since $\rightsquigarrow$ is classically equivalent to $\rightarrow$, i.e. material implication, we have $\Sigma_g \vdash e_x \rightarrow \alpha$. By the Deduction Theorem (see chapter 2) we get $\Sigma_g \cup \{e_x\} \vdash \alpha$. Thus the direction from right to left is proved.

For the other direction suppose $\Sigma_g \cup \{e_x \vdash \alpha$ and $\not\mid\!\sim_x \alpha$. Then we have $\Sigma_g \cup \{e_x\} \vdash e_x \rightarrow \alpha$ and by negative introspection ('provability of unprovability') $\mid\!\sim \neg(e_x \rightsquigarrow \alpha)$. It follows by the direction already proved that $\Sigma_g \cup \{e_x\} \vdash \neg(e_x \rightarrow \alpha)$. But this contradicts the hypothesis that $\Sigma_g \cup \{e_x\}$ is consistent. ■

Lemma 7.9 *Let the hypothesis be as in the above lemma. Suppose that x is not classical. Then $\Sigma_g \cup \{e_x\}$ is inconsistent.*

Proof. Assume $\Sigma_g \cup \{e_x\}$ is consistent. Since x is not classical there exists an α such that neither $\mid\!\sim_x \alpha$ nor $\mid\!\sim_x \neg\alpha$. It follows that

$$(1) \mid\!\sim_x \neg(e_x \rightsquigarrow \alpha$$

and

$$(2)\ \alpha \not\mid\!\sim \neg(e_x \rightsquigarrow \alpha)$$

By the above lemma we then have

$$\Sigma_g \cup \{e_x\} \vdash \neg(e_x \rightarrow \alpha)$$

By classical logic we get

$$\Sigma_g \cup \{e_x\} \vdash \alpha \rightarrow \neg(e_x \rightarrow \alpha)$$

Again by the above lemma we have

$$\mathrel{|\!\!\sim}_x \alpha \rightsquigarrow \neg(e_x \rightsquigarrow \alpha)$$

This means

$$\alpha \mathrel{|\!\!\sim}_x \neg(e_x \rightsquigarrow \alpha)$$

But this contradicts (2). It follows that $\Sigma_g \cup \{e_x\}$ is inconsistent. ■

Definition 7.10 *Let $\mathcal{A} = \langle \mathcal{X}, \mathcal{M} \rangle$ be a srongly separable M-algebra. Suppose we have a family of states $(x_i)_{i \in I}$ such that no state is orthogonal to all x_i's, equivalently if $\bigcap FP(\neg e_{x_i})$ is empty. Then we call $(x_i)_{i \in I}$ a basis of $\mathcal{A}$. We call $\mathcal{A}$ finite-dimensional if it has a finite basis.*

Definition 7.11 *Let $\mathcal{A} = \langle X, M \rangle$ be a Conjunction M-algebra. Then we call a formula φ a Kochen-Specker tautology (KS-tautology for short) for $\mathcal{A}$ if it is a (classical) tautology and $FP(\varphi) = \{0\}$.*

Lemma 7.10 *Let $\mathcal{A}$ be finite dimensional conjunction M-algebra. Let $x_i, i = 1, ..., n$ be a (finite) basis. Then we have $FP(\neg \bigwedge \neg\sigma_i) = X$.*

Proof. We need to see that $FP(\bigwedge \neg\sigma_i) = \bigcap FP(\neg\sigma_i) = \{0\}$. Assume tere is a non-zero state y in that intersection. This would mean that y is to every $x_i, i = 1, .., n$ contrary to the assumption that $x_i, i = 1, ..., n$ is a basis. ■

Lemma 7.11 *Σ_g is closed under conjunctions.*

We can now put the above lemmata together for the proof of the following theorem.

Theorem 7.3 *Let $\mathcal{A}) = \langle \mathcal{X}, \mathcal{M} \rangle$ be a finite-dimensional strongly separable Conjunction M-algebra without classical states. Then Σ_g, i.e. the global theory of $\mathcal{A}$, is (classically) inconsistent and there exists a Kochen-Specker tautology for $\mathcal{A}$.*

Corollary 7.2 *Any finite dimensional orthomodular space and thus any finite-dimensional Hilbert space of dimension at least two admits a Kochen-Specker tautology.*

Remark: We may view the above theorem as saying a bit more than just giving a sufficient condition for the existence of a Kochen-Specker tautology in certain M-algebras. We may look at this theorem as follows. Given an Implication M-algebra $\mathcal{A}$ satisfying the hyptheses of the theorem. We would then like to define a conjunction in $\mathcal{A}$ such that implication becomes definable as indicated in terms of negation and conjunction as is the case in classical logic. The theorem then says that this cannot be done in a reasonable way in the sense that any conjunction makes the global theory of $\mathcal{A}$ in the connectives $\neg$ and $\wedge$ inconsistent. So what the theorem essentially says is that M-algebras whose global theory is classically consistent do not admit a 'classical' implication.

Proof. Let $x_i, i = 1, ...n$ be a basis of $\mathcal{A}$. By lemma 7.10 we have

$$\Sigma_g \vdash \neg e_{x_i} \text{ for } i = 1, ..n.$$

Thus

$$\Sigma_g \vdash \bigwedge \neg e_{x_i}$$

On the other hand we have by lemmma 7.10

$$\neg \bigwedge \neg e_{x_i} \in \Sigma_g.$$

Therefore Σ_g is inconsistent. ■

7.6.5 Phase M-algebras

We have already seen examples of M-algebras of the sort defined above in Chapter 6. Orthomodular spaces, in particular Hilbert spaces, give rise to Conjunction M-algebras and thus to Implication M-algebras. Note that the fact that in these examples implication is given by the Sasaki hook is in view of Theorem ~refT:implication not accidental.

The following example is motivated by the concept of phase space in classical mechanics. That's why we call these M-algebras Phase M-algebras.

Definition 7.12 *Let $\mathcal{L}$ be the language of propositional logic. We call an M-algebra $\mathcal{A} = \langle \mathcal{X}, \mathcal{M} \rangle$ a Phase M-algebra if*

- *$X = X' \cup \{0\}$, where X' is a set of complete classical theories and 0 denotes the full language.*
- *M is (the language) $\mathcal{L}$*
- *$\mathcal{L}$ acts on X as follows. Let $\Sigma \in X'$ and $\alpha \in M$, then $\alpha(\Sigma) = \Sigma$ if $\alpha \in \Sigma$, else $\alpha(\Sigma) = 0$.*

Note that in the simplest case a Phase M-algebra has the form described in the above definition where $X = \{\Sigma\}$ and Σ is a complete classical theory.

But we still have to prove that Phase M-algebras are in fact M-algebras.

Lemma 7.12 *Any Phase M-algebra is an M-algebra.*

Proof. We need to verify the axioms of M-algebra. This is straightforward. Let us just sketch this for Interference. For this note the follwing two facts. We have $\Sigma \in FP(\beta$ iff $\beta \in \Sigma$. More generally, we have $\alpha(\Sigma) \in FP(\beta)$ iff not $\alpha \in \Sigma$ or $\beta \in \Sigma$. In order to prove Interference we need to verify that if $\alpha(\Sigma) \in FP(\beta)$ and $\alpha(\beta(\Sigma)) \in FP(\beta)$ then $\beta(\Sigma) \in FP(\alpha)$. The second of the above conditions says that $\alpha(\beta(\Sigma))) \in FP(\beta)$. By the above remark this is equivalent to not $\alpha \in \beta(\Sigma)$ or $\beta \in \beta(\Sigma)$. But $\beta \in FP(\beta)$ is clearly true.

■

7.6.6 Limiting case theorems

Classical mechanics is a limiting case of quantum mechanics. In which sense? In which sense do we have to 'pass to the limit from quantum mechanics' in order to get classical mechanics as the limit? There are various ways of intuitively viewing this process. One may for instance say that 'passing to the limit' means 'passing' from the presence of uncertainty relations to the absence of uncertainty relations. Another way of looking at this is to say that this process is from change of state in measurement to the absence of change in measurement Another version is to say that it is from non-commuting measurements to commuting measurements. In this section we study the way these intuitions are reflected in our framework of M-algebras.

Definition 7.13 *Let $\mathcal{A} = \langle X, M \rangle$ be an M-algebra and given a state x. We call x classical if if for every measurement α we have $\alpha(x) = x$ or $\alpha(x) = 0$. We say that $\mathcal{A}$ is commutative if all its measurements commute. We say that $\mathcal{A}$ is classical if all its measurements are classical. We say that $\mathcal{A}$ is monotonic if $\mid\!\sim_x$ is monotonic for evey $x \in X$.*

Observe that, if all measurements are classical so are all states and the other way round. So we may equivalently define an M-algebra to be classical by saying that all its states are classical.

Lemma 7.13 *If x is classical then $x \in FP(\alpha \rightsquigarrow \beta$ iff $\alpha(x) = 0$ or $\beta(x) = x$.*

Lemma 7.14 *An M-algebra is classical iff it is a Phase M-algebra.*

Theorem 7.4 *If an M-algebra $\mathcal{A} = \langle X, M \rangle$ is commutative, it is monotonic.*

Theorem 7.5 *Let $\langle X, M \rangle$ be a strongly separable Implication M-algebra and $x \in X$. Then $\mid\!\sim_x$ is monotonic iff x is classical.*

Theorem 7.6 *Let $\mathcal{A} = \langle X, M, \rangle$ be a strongly separable Implication M-algebra. Then the following conditions are equivalent:*

- *(i) $\mathcal{A}$ is commutative.*
- *(ii) $\mathcal{A}$ is monotonic.*
- *(iii) $\mathcal{A}$ is classical.*

If $\mathcal{A}$ is a Conjunction M-algebra the above conditions are equivalent to the following condition

- *(iv) $\mathcal{A}$ is a Phase M-algebra.*

Proof. We first prove that (i) implies $ii)$. Given any consequence relation $\mid\!\sim_x$. We need to show that it is monotonic. Assume $\mid\!\sim_x \beta$. This says $\beta(x) = x$. Let α be any measurement. We then have $\alpha(\beta(x)) = \alpha(x)$. Since α and β commute it follows that $\beta(\alpha(x)) = \alpha(x)$. But this says that $\alpha \mid\!\sim_x \beta$.

That (ii) implies (iii) can be seen as follows. Given any state x and any measurement α. We need to prove that either $\alpha(x) = x$ or $\alpha(x) = 0$. So assume $\alpha(x) \neq x$. It follows by 'provability of unprovaility' that $\mid\!\sim_x \neg(\sigma_x \rightsquigarrow \alpha$. By monotonicity we have $\alpha \mid\!\sim \neg(\sigma_x \rightsquigarrow \alpha)$. Thus $\alpha(x) \in FP(\neg(\sigma_x \rightsquigarrow \alpha))$.

By lemma [**?**] we have $FP(\neg(\sigma_x \rightsquigarrow \alpha)) = \{x, 0\}$ It follows that $\alpha(x) = 0$.

Clearly, (iii) implies (i).

NOT YET BUT EASY ■

7.6.7 The three faces of truth

This section is intended to be a sort of summarising reflection on chapter 7. We reflect on three notions of truth that arise naturally in connection with the material presented in chapter 7.

There is a natural notion of truth in M-algebras. Given a measurement α and a state x. Then we say hat α holds or synonymously *is true* in x if $x \in FP(\alpha)$.

The dynamic view of propositions gives rise to what earlier in this chapter we called coming true. A proposition α *comes* $trueinxifitistrue$ in $\alpha(x)$.

The probabilities given by Born's rule play the role of truth values for the coming true of propositions. This is a connection between our approach and (infinite-valued) Lukasiewicz logic. The truth values in Lukasiewicz logic are in fact values for the coming true of a proposition, 'coming truth values' rather than truth values.

The third notion of truth that we naturally encounter within the framework of M-algebras is that of self-referential truth. This notion of truth concerns metastatements. Given an Implication M-algebra $\langle x, m\rangle$ and a state $x \in X$. Then we know by lemma [**?**] that all metastatements are 'equivalent' to one of the following measurements (propositions): e_x, $\neg e_x$, $\top$, $\bot$. Comment on the peculiar action of these propositions! ELABORATE ON THIS! RESUME IT IN THE LAST CHAPTER!

Chapter 8

Aspects of Quantum Reality

In writing this chapter the authors consulted Peat's excellent book [49] "Einstein's Moon. Bell's Theorem and the Curious Quest for Quantum reality". It is possible that in this draft we used a few of Peat's formulations verbatim without quoting. We will check this and quote correctly in the final version.

8.1 General Remarks

In this Chapter we primarily address those readers who haven't had much contact with quntum mechanics yet. We report on certain typical features of the quantum world which, to these readers, may appear unfamiliar, even strange. Our intention is to convey the impression that Quantum Mechanics touches on deep issues even beyond the realm of physics.

Quantum mechanics is, in chronological order, the second great revolution in 20th century physics. The first of these two revolutions was of course Einstein's theory of relativity. The theory of relativity forced upon us the revision of long cherished views on space and time. This was a truly profound revision. It seems, however, that Quantum mechanics touches on even deeper ground and that it has an even more profound impact on the way we are forced to view the physical world. It seems that one of these issues is that of no less and no more than the nature of (physical) reality. This is the topic of a vast body of both seriously scientific and philosophical as well as popular scientific literature.

In fact it is the issue of reality that constitutes the main reason for including this chapter. This issue is of course of a philosophical, possibly even of a metaphysical nature, and, since this book is about logic and not about metaphysics, the reader may rightfully ask the question why we want to reflect on a metaphysical problem in a book on modern style logic.

Roughly, the reason is this. We said that we would be searching for the logical structures underlying quantum mechanics. Note that the emphasis is

on the term structure, and, as we said previously, we are not looking for a new deductive system that could replace classical logic as a tool of reasoning about Quantum Mechanics. Recall what Birkhoff and von Neumann say in the Introduction of [2]: "The object of the present paper is to discover what logical structures one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic". This too is our aim in this book. There is, moreover, general agreement that the search for the logical structures underlying quantum mechanics can only be successful if we leave the realm of classical logic. Somehow we will have to depart from classical logic. And, in fact, all previous attempts at constructing quantum logic departed in some way or other from classical logic.

The logical structures we want to find in quantum mechanics must reflect the way quantum mechanics departs from classical mechanics. We also said hat these structures should make precise at the logical level in which sense classical logic is a limiting case of quantum mechanics. The reader, however, can appreciate this only if he has an idea of the phenomena which, at the physical level, are characteristic of quantum mechanics and are typical of the way how quantum mechanics departs from classical mechanics. This is the purpose of this chapter.

Methodologically, the procedure is this. Taking a look at the 'strange' phenomena of the quantum world we will describe in this chapter impresses upon us certain intuitions concerning the nature of quantum mechanics and in particular the issue of reality in quantum mechanics. These intuitions will also concern the way quantum mechanics departs from Classical Mechanics, and we expect the logical structures we want to find to reflect these intuitions. They should not only reflect our intutions concerning the nature of quantum mechanics but also those concerning the relationship between classical and quantum mechanics.

What is the difference between quantum mechanics and classical mechanics? How does quantum mechanics depart from classical mechanics? We reflected on this at several points already. Essentially we said that classical mechanics is a limiting case of Quantum Mechanics. We for instance said that we may the transition from Quantum Mechanics to Classical Mechanics in various ways. We may view it as the transition from the presence of uncertainty relations to the absence of uncertainty relations, from change of state in measurement to the absence of cange of state and so on. This chapter serves to prepare the reader for another more profound way of 'passing to the limit'. This view of the transition from quantum Mechanics to classical Mechanics is concerned with the different views of reality underlying Quantum and Classical Mechanics resectively.

This is a controversial question, and it should be clear from what was said above that for our purposes we do not need to answer at it this point. But let us just make a few remarks about what it is definitely not. Take classical mechanics and relativistic mechanics. According to classical mechanics the mass of a particle, say an electron, does not change with its velocity. According to the theory of relativity it does. In this and other issues of such a sort the theories differ. But there is full agreement in both theories for instance that electrons are certain particles possessing certain properties and behaving in a certain way

under certain conditions. There is also agreement that the properties of an electron such as its position or its velocity can, at least in principle, be known through measurement. Classical and relativistic mechanics do not essentially differ with respect to their view of physical reality. They differ in their claims about the laws governing this very reality, and in this the theory of relativity constitutes a progress in comparison to classical mechanics.

As already said, the relationship between quantum mechanics and classical mechanics is of a profound nature in that it on a deep issue, namely on the issue of the very nature of (physical) reality. This insight, we hope, will lead the reader to appreciate the particular logical structures called holistic logics) which we will introduce in the next chapter.

So far, the ways of departure chosen in building 'quantum logics' consisted in abandoning or modifying certain axioms or rules of classical logic. A prominent example of this is the so called orthomodular logic where the distributive laws of classical logic are abandoned. Semantically motivated approaches attempted to view the propositional connectives of Birkhoff-von Neumann style quantum logic as intensional connectives. However, in all these attempts basic features of classical logic are retained. It seems that the 'changes' made in these modes of departing from classical logic hardly reflect the radical character of the way how quantum mechanics departs from classical mechanics.

For instance these attempts do not differ from classical logic in their style of semantics and the concept of truth. In this respect they are as committed to 'classical reality' as is classical logic. It seems that even in these attempts it is taken for granted that the language of the logic 'talks' about some external reality and the concept of truth of a formula is conceived as correspondence to some external reality. In any case we can say that in these attempts at constructing quantum logic the profound nature of the way quantum mechanics departs from classical physics is in no way reflected.

The nature of the relationship between syntax and semantics of a logic were in these attempts left untouched in the sense that it was always assumed that there exist semantic structures external to the logic itself which for instance allow the definition of truth for the logic. However different these semantic structures vary in these attempts, the dualism between syntactic and semantic representation or, roughly speaking, the dualistic relationship betweean 'logic and reality' as such is untouched. And these logics were as strongly committed to 'classical' reality as was classical mechanics. By introducing the concept of a holistic logic in the Chapter 9 we advocate a different way of departing from classical logic in this book. This way of departing from classical logic is motivated by considerations on the relationship between logic and reality. We will elaborate on this in Chapter 11.

8.2 The wave particle dualism

The first observation to shake our classical view of physical reality was de Broglies discovery of the wave-particle dualism. Elementary particles have wave

nature. An electron for instance can behave like a particle, as we would expect from classical physics, but also as a wave, which is unfamiliar from classical physics. Sometimes the electron behaves particle-like and sometimes it behaves wave-like. De Broglie even discovered a precise mathematical connection between the momentum of the electron when it is a particle and its wavelength when it is a wave. De Broglie even discovered a precise mathematical connection between the momentum of the electron when it is a particle and its wavelength when it is a wave. This is the famous de Broglie relation:

$$\lambda = h/p$$

where λ is the wavelength in case of wavelike behaviour and p is the momentum in the case of particle behaviour.

Our 'classical world view' suggests the question: Is the electron a particle or a wave? The answer is that this depends on the the particular experiment performed in order to observe the electron. There are experiments in which it behaves wave-like and there are experiments in which it behaves particle-like. It thus depends not only on the electron iteself what it 'is' but also on the observer. This is what led Bohr to the notion of complementarity. On this view the wave nature and the particle nature are mutually exclusive but *complementary* properties of one and the same physical entity. The reader will probably agree that this phenomenon is hardly reconcilable with our traditional way of looking at physical reality.

8.3 Measurement as an unseparable whole: The Copenhagen interpretation

There is a problem in quantum mechanics with ascribing definite properties to physical systems. Assume we have a quantum system in a certain state and we want to measure a physical quantity or observable E, as is the term in quantum mechanics, say its (total) energy. Then, according to quantum mechanics, this observable may not be sharp. This means that as a result of measurement we may get various values each with a certain probability. Generally, there is a discrete spectrum of values $\lambda_1, \lambda_2, ...$ that E can assume with corresponding probabilities $p_1, p_2, ...$ such that $p_i = 1$

It is important to note that these probabilities do not describe our ignorance concerning an ensemble from which we 'pick' a property at random. Rather, according to quantum mechanics, the observable E does not have definite values and it is only in the process of measurement that it assumes a certain value with a certain probability.

How can we account for this? Obviously this fact is incompatible with the view that there exists an objective physical reality having definite preexisting properties. Acording to the Copenhagen interpretation of quantum mechanics this is an expression of the fact that the quantum world consists of a set of potentialities rather than facts. These potentialities can be actualised in the

process of measurement. Heisenberg, one of the chief representatives of the Copenhagen Interpretation puts it this way: "The atoms or the elementary particles are not as real; they form a world of potentialities or possibilities rather than one of things or facts."

Describe here some more tenets of the Copenhagen interpretation.

Bohr pointed out that the process of gathering information about the microworld must, at some point, involve making an experiment in which a laboratory instrument is used. Now, the representatives of the Copenhagen inerpretation hold the view that this interaction is an indivisible whole and that it is no longer possible to analyse the parts of the two agents involved, the system to be measured and the measuring instrument in the sense that their contributionto the outcome of the interaction can be described. In Bohrs words, observer and observed system form an "indivisible, unanalysable whole" in the process of observation. *The following is a quotation from Peat "Einstein's Moon"* This holistic view of the nature of the atomic world was the key to the Copenhagen interpretation. It was something totally new in physics, although similar ideas had long been part of Eastern thought and religion. For more than two thousand years, Eastern philosophers had put forward similar views about the unity that lies between the observer and the observed. They had pointed to the illusion of breaking apart a thought from the mind that thinks the thought. Now a similar holism was entering physics.

What then is an electron, a proton or an atom? What properties does a particle have if it only manifests itself in an unanalysable interaction with a piece of apparatus? What does it mean to say that the electron *has* a certain velocity or position if every attempt to measure these proprties represent an irreducible act of interference? Indeed it becomes a major problem to speak of the electron as "having" or possessing (definite)properties. And if all the properties of a quantum object become ambiguous, then what sort of reality does it have?

Where then is atomic reality? Heisenberg suggested that the reality now lies in the mathematics. The formalism of matrix mechanics or wave mechanics works perfectly. If you want to know where atomic reality lies, then Heisenberg points to the equations; there is no hope for finding it anywhere else.

: Obviously Heisenberg wants to ascribe physical reality to, say, Hilbert space. This seems dubious to us. But what seems to us more natural is to ascribe physical reality to Hilbert space logics, i.e. to logical structures of which we may think as being 'implemented'. But this cannot be understood at this point.

Does this mean that here is no reality outside the mathemtics? about this Bohr is at his most uncompromising. " There is no quantum world", he says. "There is only an abstract quantum mechanical description." *The above is probably quoted verbatim from Peat's book "Einstein's Moon"*

8.4 Are there "elements of reality"? EPR and non-locality

Here we try to give the gist of the EPR-paradox in an informal way.

We give here a rough description of the EPR paradox. In 1935 Einstein, Podolsky and Rosen published a now famous paper entitled "Can the quantum mechanical description of physical reality be considered complete?"[13]. In this paper an ingenious thought experiment is presented which seemed to shake the newly established edifice of quantum mechanics. This was after the great intellectual struggle between Einstein and Bohr over the foundations of quantum mechanics. In that struggle Bohr was generally considered winner. But now, after the EPR paper, Einstein had reappearsd on the stage with a seemingly devastating argument against quantum mechanics. And, in fact, it took Bohr some time to recover from this blow until he managed to produce a defence against the attack.

We try to present the EPR argument in a nutshell. In this we are aware of the fact that we cannot, in this context, do justice to all its subtleties, let alone Bohr's reply.

Imagine a particle at rest, i.e. with momentum zero, being split into electrons moving away in opposite directions. Now, assume electron 1 and electron 2 are far apart from each other and there is no physical interaction between them. We now assume that the position of electron 1 is measured. Once position of electron 1 is known, the position of electron 2 is known too. So, if we want to know the position of electron 2, it suffices to measure the position of electron 1. Now it is reasonable to assume that performing a measurement on electron 1 does not in any way disturb electron 2. The term used by EPR is "without in any way disturbing". So we can measure the position of electron 2 "without in any way disturbing" it. Put differently, we can predict with certainty the position of electron 2 without in any way desturbing it. Now, EPR say that if a quantity of a system can be predicted with certainty without in any way disturbing it, then this quantity constitutes an "element of reality". So the position of particle 2 constitutes an "element of reality".

The point of the argument is now this. Instead of measuring position of electron 2 by performing a measurement of momntum on electron 1 we may measure momentum of electron 2 by performing a measurement of momentum on electron 1. So, again we can predict the momentum of electron 2 without in any way disturbing it. Therefore the momentum of electron 2 constitutes an "element of reality". But now note that according to quantum mechanics a physical system cannot simultaneously possess both a sharp position and a sharp momentum, which according to EPR are both elements of reality The conclusion EPR now draw is that Quantum Mechanics does not provide a complete description of reality in that it does not capture all elements of reality. We cannot now go into a detailed discussion of the EPR thought experiment with all its subtleties. But it is obvious from the above that the EPR argument contains at least two tacit assumptions. The first assumptions concerns the existence of

separate elements of reality, and the second assumption is implicit in the term "without in any way disturbing". The first assumption reflects a picture of a fragmented reality, which consists of separate elements, and the second assumption is what is nowadays called nonlocality. If, therefore, quantum mechanics is a correct description of reality, which is widely believed nowadays, then either one or even both of these assumptions must be false. Both assumptions concern the nature of reality. So we can say that if Quantum Mechanics does provide a complete description of physical reality and the EPR argument is thus false, then the EPR argument is false because it rests on a wrong view of reality.

But is there any way of proving that this view of reality is wrong? The answer is yes. It is given by *Bell's Theorem*. Bell's theorem provides us with the means of experimentally testing whether physical reality is local or not. This test has been performed in several ingenious experiments by Aspect. And the experimental finding is: *Physical reality is non-local*. Bell's theorem thus permitted us to unveil a feature of physical reality which was undreamed of until it was, on the basis of Bell's theorem, discovered experimentally. This is the reason why Henry Stapp, reputed American physicist and leading expert on the foundations of quantum mechanics, called Bell's Theorem the greatest scientific discovery ever.

What is Bell's theorem? We need not go into the details in order to understand how it opened up the possibility of experimentally testing whether reality is local or not. Bell's Theorem is a statement of the form: "If reality is local, then A". The point is that statement A can be *experimentally tested*. Experiment yields: Not A. We conclude: Reality is not local. But there is more to Bell's Theorem. Namely, experiment not only yields NOT A, but it yields B. Again, B can be experimentally verified, and B is what quantum mechanics predicts. A triumph of quantum mechanics!

Here we will, in the final version, give a more detailed account of Bell's Theorem.

What was Bohr'a answer to EPR ? Certainly he did not know Bell's theorem yet. But, essentially, Bohr argued that the EPR argument suffered from a fundamental flaw, namely that is was based on a wrong view of (physical) reality. In Bohm's words: "He (Bohr) argued that in the quantum domain the procedure by which we analyse classical systems into interacting parts breaks down, for whenever two entities combine to form a single system (even if only for a limited period of time) the process by which they do this is not divisible" The gist of Bohr's reply is that the two electrons form an indivisible whole to which our method of 'fragmenting' is not applicable.

8.5 Bohm on wholeness and his experiment with language

David Bohms creative life was in its many facets devoted to the puzzle of quantum mechanics. The chain of his thinking contains brilliant ideas all of which

became highly significant and form essential parts of the literature on the foundations of quantum mechanics.

His views on physical reality with which he had come up after almost half a century of dedicated intellectual work on the problem of understanding quantum mechanics culminated in his book "Wholeness and the Implicate Order". In this book Bohm consistently argues for a holistic world view in order to account for the puzzles of quantum reality. This holistic world view which should replace the fragnentary world view typical of Western thought and of modern Western science such as classical physics. In the first chapter entitled "Fragmentation and Wholeness" he describes the act of observation (measurement) in quantum mechanics as follows: "One can no longer maintain the division between the observer and the observed (which is implicit in the atomistic view that regards each of these as separate aggregates of atoms). Rather, both observer and observed are merging and interpenetrating aspects of one reality, which is indivisible and unanalysable." Another quotation:" ...relativity and quantum theory agree in that they both imply the need to look on the world as an undivided whole, in whih all parts of the universe, including the observer and his instruments merge and unite in one totality. In this totality, the atomistic form of insight is a simplification and an abstraction, valid only in some limited context." Bohm then suggests to view reality as an "Undivided Wholeness in Flowing Movement". This view put forward by an outstanding representative of modern Western science is undoubtedly reminiscent not only of Eastern thought but also of Heraclitus philosophy of the world as being in in permanent flux. The idea that our language, i.e. the language we use in everyday life and also in classical physics, may not be the appropriate language for the quantum world has already been put forward by Heisenberg in his "Physics and Philosophy". Heisenberg holds that the puzzling nature of quantum mechanics is due to the fact that the structure of our languge does not fit in the pattern of quantum reality. He does not, however, make any attempt to actually describe the linguistic structures that might conform to quantum mechanics. It is in particular the subject-predicate structure noun phrase - verb phrase) of the sentence that reflects the fragmented nature of our world view. The noun phrase, in particular the definite noun phrase 'denote' objects of an external world, verb phrases 'denote' properties, predicates, transitive verbs 'denote' relations between objects etc.

The metaphor Bohm uses in "Wholeness and the implicate order" is that of the of a hologram. The word being of Greek origin denotes an instrument 'writing the whole'.

In order to understand the metaphor of the hologram in Bohm's thinking we need to know a bit about the nature of a hologram without, however, having to go into the technical details of actually constructing such an optical instrument. Let us just say this. A hologram is commonly known as a three-dimensional photograph made with the the help of a laser. Using laser light, an interference pattern on a photographic plate is created. The developed film is then illuminated again by laser light. Then a three-dimensional image of the photographed object appears. It is not the impression of three-dimensionality, however, that

is most striking about a hologram. Rather it is the following. When we illuminate just a part of the photographic plate, however small, it is, though smaller, still the (whole) image of the *whole object* that appears. So every part of the hologram encodes the whole information possessed by the whole object. Technically, this effect is explained by the wave nature of light and the particular interference effect creating the pattern on the photographic plate.

The main difference between the hologram and the familiar instrument of a *lens* is this. A lens is an optical instrument creating an image of an object in such a way that that the parts of the objects correspond to the parts of the image in a one-to-one way. But, due to the wave properties of light, even this is true only approximately. Bohm remarks that thus the case of the lens may be regarded as the limiting case of the hologram. Fragmentation is so to speak the limiting case of wholeness. This intuition is reflected in the precise logical framework of Chapter as the Limiting case Theorem.

One of the characteristic feature of the hologram is that it is mirrored (encoded) in all its parts. Our view of reality should, according to Bohm, be holistic in the sense that there is no fragmentation. Rather, the 'parts' are to reflect and encode the 'whole'. This is also strikingly reminiscent of what Leibniz says about the monads, his incorporeal 'atoms of reality'. These monads, Leibniz says, 'mirror' each other. The whole world is thus mirrored in each of its parts.

But for Bohm this is not just a good metaphor. Rather, in the second chapter of "Woleness and the Implicate Order" entitled "The rheomode- an experiment with language and thought" he takes this picture seriously as the basis for constructing new linguistic structures which he thinks are more appropriate to the type of reality suggested by quantum mechanics. What is the rheomode? If our language with is typical subject predicate sentence structure is not the language appropriate for the quantum world, what then does the proper language look like? It would, as Bohm correctly points out, not be practicable to construct a new language having an entirely new structure appropriate for the quantum world. Instead, Bohm proposes a new mode of language similar to that of indicative, imperative, subjunctive He says:"... will now consider a mode in which movement is to be taken as primary in our thinking and in which this motion will be incorporated into the language structure by allowing the verb rather than the noun to play a primary role... For the sake of convenience we shall give this mode a name, i.e. the rheomode (rheo is from a Greek verb, meaning to flow). At least in the first instance the rheomode will be an experiment with language, concerned mainly with trying to find out whether it is possible to create a new structure that is not so prone towards fragmentation as is the present one."

In order to give a flavour of Bohm's rheomode experiment let us take a look at his discussion of terms like 'relevance', 'relevant'...Bohm introduces the rheomode with a discussion of the issue of relevance.

Here Bohm proposes to consider verbs such as 'to levate' or 'relevate', which are not part of language so far or any longer, because they may have dropped out of language , as basic. Why does he choose the notion of relevance as the starting point for hisinqiry into the rheomode? When using the term 'relevant' we focus on a whole bunch of things. First, clearly, we focus on something wec

consider relevant, say a sentence uttered in a discussion. But we also focus on the process of thought or perception as a result of which this sentence appears relevant. Moreover, its relevance depends on the context, which can change and does change. So, in making statements about relevance we refer to an integrated whole involving language, perception extralinguistic contexts, and this whole is in flux. The boundaries between relevance and irrelevance are not sharp as the structure of language with the dominating nouns'relevance' and 'irrelevance' would suggest. This, says Bohm, is a case for the rheomode. heproposes to introduce the verb 'to relevate' into the language Which he says should mean " to lift a certain context into attention again, for a particular context as indicated by thought and language". 're-levant' then denotes the state of being 're-levated'. Note that 're' means again. "It implies time and similarity (as well as difference)"

"So when relevance or irrelevance is communicated, one has to understand that this is not a hard and fast division between opposing categories but rather, an expression of an ever-changing perception, in which it is possible, for the moment, to see a fit or non-fit between the content lifted into attention and the context to which it refers"

8.6 Informal reflections

The above remarks were concerned with the issue of reality in quantum mechanics. As already mentioned, this could not be in any way exhaustive. We scratched the surface only. We hope, however, that we succeeded in conveying the impression to the reader that understanding the "mystery of quantum mechanics" demands a revision of our views on the nature of (physical) reality. In this endeavour, we will have to depart from the world view that underlies not only everyday life but also Newtonian and relativistic physics. What we come across all the time when discussing the issue of reality in quantum mechanicis is the need to revise our familiar view of a fragmented world that can be separated into "elements of reality" in favour of a more holistic view of reality. This is a profound revision.

In this book we try to cast light on the "mystery of quantum mechanics" from the point of view of logic. In this we will have to depart from classical logic. The question is what this way of departure will have to look like. The dilemma is, roughly, this. Modern logic, in particular the modern version of classical logic, with all its merits and great achievements, is still based, in particular with regard to its semantics, on the fragmented view of reality which underlies classical physics and which, in whatever sense, needs revision in quantum mechanics. We therefore must ask ourselves the question whether any way of departing from classical logic that does not in some way reflect this profound revision can be successful. It was the purpose of this chapter to impress onto the reader the idea that in order to make logic fruitful for the enterprise of trying to understand quantum mechanics a profound revision is necessary. This revision must concern the issue of reality in logic in the same way as quantum mechanics forces us to

revise our view of physical reality of classical mechanics. We will in our way of departing from classical logic be guided by this idea.

This book takes at least part of its inspiration from Bohm's book. As already pointed out we view the logical structures we set out to discover in this book as a sort of mirrors of quantum reality. In Bohm's experiment it is a mode of language in which the dynamic and holistic aspects of quantum reality are reflected. In our approach it is logic, more precisely certain logical structures that are to play this role. Recall our treatment of propositions in M-algebras. There, propositions are viewed dynamically rather than statically, in accordance with Bohm's picture of a world in flux. In Chapter 9 on holistic logics it is the intuition of wholeness and interconnectedness which is reflected in a similar way.

Chapter 9

Holistic Logics

In this chapter we introduce, as in chapter 6, certain structures which are abstractions from Hilbert space. In Chapter 6 the focus was on propositions. The structures we introduced, namely M-algebras, were designed to capture the way how propositions act on states. States were, there, primitive notions. In chapter 7 we went one step further. We focused on the nature of the states themselves. In all this, however, we did not leave the framework of M-algebras. In this chapter we also abstract from this. The central concept we introduce in this chapter is that of a *holistic logic*. Our motivation for choosing this term is given by our intuitive reflections on the concept of a state in a dynamic logical framework, see Chapter 6. We saw that states in such a framework must encode other states. This intuition is perfectly reflected in the logical structures we call holistic logics. Loosely speaking, in such systems "everything is encoded in (almost) everything".

Throughout this chapter we assume a language Fml of propositional logic as introduced in Chapter 2.

9.1 Consequence Revision Systems

9.1.1 Formal Motivation: the Lindenbaum algebra viewed as an operator algebra

In order to motivate the concepts we are going to introduce we start with an observation from classical logic. Recall that $\vdash$ denotes the consequence relation of classical logic. Given a formula α, we may form a new consequence relation $\vdash_\alpha$ as follows: $\beta \vdash_\alpha \gamma$ iff $\alpha \wedge \beta \vdash \gamma$. We get a class of consequence relations $\mathcal{C} = \{\vdash_\alpha \mid \alpha \in Fml\}$. By the Deduction Theorem of classical logic (see the Chapter 2) we have $\beta \vdash_\alpha \gamma$ iff $\vdash_\alpha \beta \to \gamma$ for all $\vdash_\alpha \in \mathcal{C}$. We say that $\to$, i.e. material implication, is an internalising connective for $\mathcal{C}$. Again, given $\alpha \in Fml$ and $\vdash_\beta \in \mathcal{C}$, we may form the consequence relation $\vdash_{\alpha\wedge\beta}$. Thus every $\alpha \in Fml$ induces an operator $\overline{\alpha} : \mathcal{C} \to \mathcal{C}$. We have $\overline{\alpha} = \overline{\beta}$ iff α and β are classically

equivalent. It is readily verified that the class of operators is partially ordered by: $\overline{\alpha} \leq \overline{\beta}$ iff $\alpha \vdash \beta$. Moreover, it is routine to verify that this structure forms a Boolean algebra isomorphic to the Lindenbaum-Tarski algebra of classical logic. Observe that $\overline{\beta}\overline{\alpha} = \overline{\alpha}$ iff $\alpha \vdash \beta$. This is our motivating example of what we will call a *consequence revision system.* Its main ingredients are a *class of consequence relations* $\mathcal{C}$, a function $F : Fml \times \mathcal{C} \to \mathcal{C}$ and a connective, which in this case is material implication, which is an internalising connective for all consequence relations of $\mathcal{C}$, i.e. we have $\vdash_\alpha \beta \to \gamma$ iff $\beta \vdash_\alpha \gamma$ for any α. The structure of interest is the triple $\mathcal{L} = \langle$C,F, $\to\rangle$.

There is a straightforward generalisation of the above consideration. We could have started with some consistent set of formulas Σ and the consequence relation $\vdash_\Sigma$ defined by: $\alpha \vdash_\Sigma \beta$ iff $\Sigma \cup \{\alpha\} \vdash \beta$ and would by the same procedure as above have arrived at the structure $\mathcal{L}_\Sigma = \langle \mathcal{C}_\Sigma, \mathcal{F}_\Sigma, \to \rangle$. Note that, by the Deduction Theorem of classical logic, material implication is still the internalising connective in this more general case.

9.1.2 Consequence relations

We shall, in this chapter, be concerned with classes of consequence relations and must therefore consider conditions these consequence relations are supposed to satisfy. These conditions go beyond those stated in Chapter 2. But one should note that all these conditions do hold in Hilbert space, as we will see in the next chapter. Moreover, we assume in this chapter a language with all the usual propositional connectives, i.e. $\neg$, $\wedge$, $\vee$, $\to$.

We denote the universal (inconsistent) universal consequence relation by 0. We assume that for the consequence relations we consider this is equivalent to the existence of a formula α such that $\mid\sim \alpha$ and $\mid\sim \neg\alpha$. Any class of consequence relations considered is assumed to contain 0. That means we assume for any $\mid\sim \neq 0$ that for no $\alpha \in Fml$ we have $\mid\sim \alpha$ and $\mid\sim \neg\alpha$.

Given a class $\mathcal{C}$ of consequence relations. Then we write $\alpha \mid\sim_\mathcal{C} \beta$ iff $\alpha \mid\sim \beta$ for every $\mid\sim \in \mathcal{C}$. We say $\alpha \equiv_\mathcal{C} \beta$ if $\alpha \mid\sim_\mathcal{C} \beta$ and $\beta \mid\sim_\mathcal{C} \alpha$.

Minimal Conditions 1

Let us for the sake of convenience again mention here the minimal conditions of Chapter 2 which are generally imposed on consequence relations.

Reflexivity

$$\alpha \mid\sim \alpha$$

Cut

$$\frac{\alpha \wedge \beta \mid\sim \gamma,\ \alpha \mid\sim \beta}{\alpha \mid\sim \gamma}$$

Restricted Monotonicity

$$\frac{\alpha \mid\sim \beta,\ \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$$

In the paper by Kraus–Lehmann–Magidor [34] these three conditions are, as suggested by Gabbay in [**?**], considered to be the minimal conditions a consequence relation should satisfy.

For a given consequence relation $\mid\!\sim$ define

$$\alpha \equiv \beta \text{ iff } \alpha \mid\!\sim \beta \text{ and } \beta \mid\!\sim \alpha$$

Minimal Conditions 2

Moreover, we impose the following conditions on the special consequence relations studied in this book.

$$\alpha \equiv \neg\neg\alpha$$

$$\top \equiv \alpha \vee \neg\alpha$$

$$\bot \equiv \alpha \wedge \neg\alpha$$

$$\alpha \wedge \beta \mid\!\sim \alpha$$

$$\alpha \wedge \beta \mid\!\sim \beta$$

$$\alpha \mid\!\sim \alpha \vee \beta$$

$$\beta \mid\!\sim \alpha \vee \beta$$

$$\mid\!\sim \alpha \vee \neg\alpha$$

$$\alpha \mid\!\sim \top$$

$$\bot \mid\!\sim \alpha$$

$$\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$$

$$\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$$

The conditions we imposed so far are 'local' in nature in the sense that they are imposed separately on every single consequence relation belonging to the class considered. We, moreover, impose the following conditions which have a global character in the sense that they are related to the class $\mathcal{C}$ as a whole.

$$\frac{\alpha \mid\!\sim_{\mathcal{C}} \gamma, \beta \mid\!\sim_{\mathcal{C}} \gamma}{\alpha \vee \beta \mid\!\sim_{\mathcal{C}} \gamma}$$

$$\frac{\alpha \mid\!\sim_{\mathcal{C}} \beta}{\neg\beta \mid\!\sim_{\mathcal{C}} \neg\alpha}$$

The reason for imposing these conditions is that we want to have the algebraic structures arising from these logical structures to have certain desirable properties that are actually fulfilled in the case of the concrete structures arising in connection with Quantum Mechanics. So, in contrast to the last two chapter we want the relevant algebraic structures to be lattices. This brings us closer to Hilbert space where the algebraic structures relevant from the logial point of view are in fact lattices.

9.1.3 The Concept of a Consequence Revision System

Definition 9.1 *Let Fml be a class of formulas as described above and let $\mathcal{C}$ be a class of consequence relations over Fml satisfying the conditions described. Let F be a function*

$$F : Fml \times \mathcal{C} \to \mathcal{C}.$$

Then we say that F is an action on $\mathcal{C}$ iff for every $\mid\!\sim \in \mathcal{C}$ and $\alpha, \beta \in Fml$ the following conditions are satisfied.

$$\textit{(i)}\ F(\top, \mid\!\sim) = \mid\!\sim$$

$$\textit{(ii)}\ F(\alpha, \mid\!\sim) = 0 \ \textit{iff} \mid\!\sim \neg\alpha$$

$$\textit{(iii)}\ F(\beta, F(\alpha, \mid\!\sim)) = F(\alpha, \mid\!\sim) \ \textit{iff}\ \alpha \mid\!\sim \beta$$

If F is an action on $\mathcal{C}$, we call the pair $\langle \mathcal{C}, F \rangle$ a consequence revision system (CRS).

Note that by $\mid\!\sim \alpha$ we mean $\top \mid\!\sim \alpha$. For a given class $\mathcal{C}$ of consequence relations call the formulas α and β $\mathcal{C}$-equivalent, in symbols $\alpha \equiv_{\mathcal{C}} \beta$, if for every $\mid\!\sim \in \mathcal{C}$ we have $\alpha \mid\!\sim \beta$ and $\beta \mid\!\sim \alpha$.
Remark: We are aware of the fact that the way we use the term revision in the above definition does not fully capture the way it is used in traditional revision theory. If at all, the action of formulas on consequence relations as defined above represents a simple type of revision. Condition (ii) above says that given a consequence relation $\mid\!\sim$ and a formula α which is inconsistent with $\mid\!\sim$ then the result of 'revising' $\mid\!\sim$ by α is the inconsistent consequence relation. The corresponding case in traditional revision thory is that of a theory T and a formula α inconsistent with T. The result of revising T by α usually denoted by $T * \alpha$ is then, according to according to traditional revision theory, not necessarily the inconsistent theory. Since, however, in our most important examples, namely those arising from Hilbert spaces, we are concerned with a process which, in the intuitive sense, deserves to be called revision, we freely use the term revision. Every $\alpha \in Fml$ induces a (revision) operator on $\mathcal{C}$

$$\overline{\alpha} : \mathcal{C} \to \mathcal{C}$$

via

$$\overline{\alpha} \mid\!\sim =: F(\alpha, \mid\!\sim)$$

For $\overline{\alpha} \mid\!\sim$ we will also write $\mid\!\sim_\alpha$.

Denote the class of these operators by $\overline{Fml}$. We have $\overline{\alpha} = \overline{\beta}$ iff $\alpha \equiv_{\mathcal{C}} \beta$.

$\overline{Fml}$ has the structure of a (multiplicative) semigroup the multiplication being

$$\overline{\alpha}\overline{\beta} = \overline{\alpha} \circ \overline{\beta}$$

Lemma 9.1 *For any $\alpha \in Fml$ we have $\overline{\alpha} \circ \alpha = \overline{\alpha}$.*

Proof. By *Reflexivity* we have $\alpha \mid\!\sim \alpha$ for every $\mid\!\sim \in \mathcal{C}$. Thus the claim follows by conditions (iii) of the definition of an action. ▮

We have a natural partial ordering on $\overline{Fml}$, namely $\overline{\alpha} \leq \beta$: iff $\overline{\beta}\overline{\alpha} = \overline{\alpha}$

Lemma 9.2 *Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then for any $\mid\!\sim \in \mathcal{C}$ the following conditions are equivalent*

- *(i)* $\mid\!\sim \alpha$
- *(ii)* $\mid\!\sim_\alpha = \mid\!\sim$
- *(iii) There exists a* $\mid\!\sim_1 \in \mathcal{C}$ *such that* $\mid\!\sim_{1,\alpha} = \mid\!\sim$

Proof. For the equivalence of (i) and (ii) observe first that $\mid\!\sim_{\top,\alpha} = \mid\!\sim_\alpha$. By condition (iii) of the definition of an action we have that $\top \mid\!\sim \alpha$ iff $\mid\!\sim_\alpha = \mid\!\sim_{\top,\alpha} = \mid\!\sim_\top = \mid\!\sim$. Clearly, (ii) implies (iii). In order to show that (iii) implies (ii) suppose $\mid\!\sim_{1,\alpha} = \mid\!\sim$. Note that by *Reflexivity* we have $\alpha \mid\!\sim_1 \alpha$. Then it follows by condition (iii) of the definition of an action that $\mid\!\sim_\alpha = \mid\!\sim$. ▮

Lemma 9.3 $\alpha \mid\!\sim \beta$ *iff* $\mid\!\sim_\alpha \beta$*,*

Proof. Suppose $\alpha \mid\!\sim \beta$. By condition (iii) of the defininition of an action this is equivalent to $\mid\!\sim_{\alpha,\beta} = \mid\!\sim_\alpha$. By (i) of the above lemma this means that $\mid\!\sim_\alpha \beta$. ▮

It follows by the above two lemmas that $\mid\!\sim_\alpha = \mid\!\sim_\beta$ implies $\alpha \equiv \beta$, i.e. $\alpha \mid\!\sim \beta$ and $\beta \mid\!\sim \alpha$. We see that $\overline{\alpha} = \overline{\beta}$ iff $\alpha \equiv_{\mathcal{C}} \beta$.

Definition 9.2 *Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then define the proposition $[\alpha]$ as follows.*

$$[\alpha] =: \{\mid\!\sim \mid\ \mid\!\sim \alpha\}$$

We denote the class of propositions of $\langle \mathcal{C}, F \rangle$ by Prop.

It is routine to verify the statements made in the following lemma.

Lemma 9.4 *Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then*

$$\overline{\alpha} \leq \overline{\beta} \textit{ iff } [\alpha] \subset [\beta]$$

$$\overline{\alpha} = \overline{\beta} \textit{ iff } [\alpha] = [\beta]$$

$$\alpha \mid\!\sim_{\mathcal{C}} \beta \textit{ iff } [\alpha] \subset [\beta]$$

$$\alpha \equiv_{\mathcal{C}} \beta \textit{ iff } [\alpha] = [\beta]$$

$$\overline{\alpha} \leq \overline{\beta} \textit{ iff } \overline{\neg\beta} \leq \overline{\neg\alpha}$$

$$[\alpha] \subset [\beta] \textit{ iff } [\neg\beta] \subset [\neg\alpha]$$

The conditions we imposed on the consequence relations guarantee that the following holds. IN THE FINAL VERSION WE WILL MAKE THIS EXPLICIT.

Lemma 9.5 *For any CRS both $\langle \overline{Fml}, \leq \rangle$ and $\langle Prop, \subset \rangle$ are lattices. For $\overline{\alpha}, \overline{\beta} \in \overline{Fml}$ and $[\alpha], [\beta] \in Prop$ the greatest lower bounds are $\overline{\alpha \wedge \beta}$ and $[\alpha \wedge \beta]$ respectively.The lowest upper bounds are given by $\overline{\alpha \vee \beta}$ and $[\alpha \vee \beta]$ respectively.*

Proof. *The proof is routine. We will give it in full in the final version* ▮

Given a CRS $\langle \mathcal{C}, F \rangle$. Then define unary operations $^* : \overline{Fml} \to \overline{Fml}$ and $^* : Prop \to Prop$ as follows.

$$\overline{\alpha}^* =: \overline{\neg \alpha}$$

and

$$[\alpha]^* =: [\neg \alpha]$$

Note that in view of Lemma 11 these operations are well defined. Moreover, we define a mapping $\psi : \overline{Fml} \to \ Prop$ by

$$\psi(\overline{\alpha}) = [\alpha]$$

again, by Lemma ? this mapping is well defined. It is routine to verify the following theorem which bears an analogy to the well known fact that in Hilbert space the lattice of projectors and the lattices of closed subspaces are isomorphic (orthomodular) lattices.

Theorem 9.1 *Let $\langle \mathcal{C}, F \rangle$ be a CRS. Then*

- $\langle Fml, \leq, ^* \rangle$ *and* $\langle Prop, \subset, ^* \rangle$ *are ortholattices.*
- ψ *is an isomorphism between ortholattices.*

Proof. *This is one of the several proofs which are routine and are not yet included in this draft. In the final version they will be given in full.* ▮

9.1.4 The Concept of an Internalsing Connective

We now define the concept of an *internalising connective* which, essentially, is already familiar from Chapter 7 in the context of Consequence Revision Systems. From the logical point of view the concept of an internalising connective is the link between the object level and the meta level. Note that whenever we use the term connective we mean a connective definable by the usual propositional connectives in the following sense. We say that $\alpha\beta$ is definable if there exists a formula of propositional logic $\varphi(p, q)$ with exactly two propositional variables such that the formula $\alpha \rightsquigarrow \beta$ is the result of uniformly substituting α in φ for p and β for q. We say that φ defines $\rightsquigarrow$. Given two connectives $\rightsquigarrow_1$ and $\rightsquigarrow_2$

defined by φ_1 and φ_2 respectively. Then we say that $\rightsquigarrow_1$ and $\rightsquigarrow_2$ are classically equivalent if φ_1 and φ_2 are classically equivalent.

As already said, material implication is internalising for the classical consequence relation $\vdash$. This follows from the Deduction Theorem of classical logic. So the fact that a given consequence relation -classical (monotonic) or non-classical (non-monotonic)- admits an internalising connective may be viewed as a sort of generaliseed Deduction Theorem.

Consider for instance the Sasaki hook $\rightsquigarrow_s$ which is defined by $\varphi(p,q) =: \neg p \vee (p \wedge q)$. This says that $\alpha \rightsquigarrow_s \beta$ is just short for $\neg\alpha \vee (\alpha \wedge \beta)$. Obviously the Sasaki hook is classically equivalent to material implication.

Definition 9.3 *Let $\mid\!\sim$ be a consequence relation and $\rightsquigarrow$ a connective such that $\alpha \mid\!\sim \beta$ iff $\mid\!\sim \alpha \rightsquigarrow \beta$ we say that $\rightsquigarrow$ is an internalising connective for $\mid\!\sim$. Given a CRS $\langle \mathcal{C}, F\rangle$. Then we say that $\rightsquigarrow$ is an connective. Then we say that $\rightsquigarrow$ is an internalising connective for $\langle \mathcal{C}, F\rangle$ iff $\rightsquigarrow$ is an internalising connective for all $\mid\!\sim \in \mathcal{C}$.*

Lemma 9.6 *Let $\langle \mathcal{C}, F\rangle$ be a CRS and let $\rightsquigarrow$ be an internalising connective for $\langle \mathcal{C}, F\rangle$. Then the following holds.*

- *(i) $\alpha \mid\!\sim (\beta \rightsquigarrow \gamma)$ iff $\beta \mid\!\sim_\alpha \gamma$*
- *(ii)$\{\mid\!\sim \mid \alpha \mid\!\sim \beta\}$ is a proposition, namely $[\alpha \rightsquigarrow \beta]$*

Proof. By **??** we have $\alpha \mid\!\sim (\beta \rightsquigarrow \gamma)$ iff $\mid\!\sim_\alpha (\beta \rightsquigarrow \gamma)$. Since $\rightsquigarrow$ is internalising, this is equivalent to $\beta \mid\!\sim_\alpha \gamma$. This proves (i).
(ii) follows from the fact that $\rightsquigarrow$ is internalising. ∎

Note that in case we have an internalising connective $\rightsquigarrow$ the process of revision can be described very simply as follows. Revise the consequence relation $\mid\!\sim$ by α so as to get $\mid\!\sim_\alpha$. Then γ can be proved from β in $\mid\!\sim_\alpha$ iff $\beta \rightsquigarrow \gamma$ can be proved from α in $\mid\!\sim$.

Given a class of consequence relations $\mathcal{C}$ and two connectives $\rightsquigarrow_1$ and $\rightsquigarrow_2$. We then say that $\rightsquigarrow_1$ and $\rightsquigarrow_2$ are $\mathcal{C}$-equivalent iff for all formulas $\alpha, \beta \in Fml$ we have $\alpha \rightsquigarrow_1 \beta \equiv_{\mathcal{C}} \alpha \rightsquigarrow_2 \beta$.

Lemma 9.7 *Let $\langle \mathcal{C}, F\rangle$ be a CRS. Then any two internalising connectives for $\langle \mathcal{C}, F\rangle$ are $\mathcal{C}$-equivalent.*

Proof. Let $\rightsquigarrow_1$ and $\rightsquigarrow_2$ be two internalising connectives for $\langle \mathcal{C}, F\rangle$. By symmetry it suffices to prove that $\alpha \rightsquigarrow_1 \beta \mid\!\sim_{\mathcal{C}} \alpha \rightsquigarrow_2 \beta$. So let $\mid\!\sim$ be any element of $\mathcal{C}$ such that $\mid\!\sim \alpha \rightsquigarrow_1 \beta$. Since $\rightsquigarrow_1$ is internalising, we have $\alpha \mid\!\sim \beta$ and, since $\rightsquigarrow_2$ is internalising, $\mid\!\sim \alpha \rightsquigarrow_2 \beta$. ∎

The above lemma says that the action 'determines' the internalising connective modulo $\mathcal{C}$-equivalence. The next lemma states a sort of converse for this, namely that the internalising connective 'determines' the action.

Lemma 9.8 *Let $\langle \mathcal{C}, F_1\rangle$ and $\langle \mathcal{C}, F_2\rangle$ be CRS and let $\rightsquigarrow$ be a connective which is internalising for both. Then we have $F_1 = F_2$.*

Let us now come to a crucial point. In principle, the concept of an action of formulas on a class of consequence relations can serve a useful purpose. It may serve as a vehicle for studying the interplay between properties of the operator algebra on the one hand and properties of the class of consequence relations on the other. Generally, properties of the former type are algebraic in nature, whereas properties of the latter type are logical in nature. The orthocomplemented lattice of operators and thus the lattice of propositions may have the algebraic property of being orthomodular and we may ask the question what is the 'logical' counterpart of that algebraic property. The situation we have is, by the way, familiar from various branches of mathematics. It is for instance a familiar technique to study the algebraic structure of certain groups via their action on certain spaces making use of geometrical or topological properties of these spaces. As to the concept of an orthomodular lattice note that this is a dominant concept in virtually all approaches to quantum logic. It is so to speak the quantum logical counterpart of the concept of a Boolean algebra in classical logic. The most prominent examples are the lattices of closed subspaces of orthomodular spaces and in particular Hilbert spaces.

As already mentioned the following connective $\leadsto_s$ is called *Sasaki hook*

$$\alpha \leadsto_s \beta =: \neg\alpha \vee (\alpha \wedge \beta)$$

Theorem 9.2 *Let $\langle \mathcal{C}, F\rangle$ be a CRS such that for any $\mid\!\sim \in \mathcal{C}$, $\mid\!\sim (\alpha \leadsto_s \beta)$ implies $\alpha \mid\!\sim \beta$ and let $\leadsto$ be an internalising connective for $\langle \mathcal{C}, F\rangle$. Then $\langle \overline{Fml}, \leq, ^* \rangle$ and thus $\langle Prop, \subset, ^* \rangle$ are orthomodular lattices and $\leadsto$ is $\mathcal{C}$-equivalent to $\leadsto_s$. If $\leadsto_s$ is $\mathcal{C}$-equivalent to $\rightarrow$, i.e. material implication, then the above lattices are Boolean algebras.*

Proof. In view of 9.1 it suffices to prove orthomodularity. We first show that for any $\mid\!\sim \in \mathcal{C}$

$$(1)\ \alpha \wedge (\alpha \leadsto \beta) \mid\!\sim \beta$$

By Lemma ? it suffices to show that $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)} \beta$. By Lemma ? we have $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)} \alpha \wedge (\alpha \leadsto \beta)$ and thus $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)} \alpha$ and $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)} \alpha \leadsto \beta$. Moreover, since $\leadsto$ is internalising, we have $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta),\alpha} \beta$. But $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta),\alpha} = \mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)}$, since $\mid\!\sim_{\alpha\wedge(\alpha\leadsto\beta)} \alpha$. Now (1) is proved.
It follows that

$$(2)\ \overline{\alpha} \wedge \overline{\alpha \leadsto \beta} \leq \overline{\beta}$$

We now prove that the operator $\overline{\alpha \leadsto \beta}$ has the following property.

$$(3)\ \overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma} \text{ implies } \overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}.$$

For this we have to use that every $\mid\!\sim \in \mathcal{C}$ satisfies *Cut*. Assume $\overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma}$ and let $\mid\!\sim \in \mathcal{C}$ be such that $\mid\!\sim \alpha \leadsto \beta$. We then have $\alpha \wedge \beta \mid\!\sim \gamma$ and, since $\leadsto$ is internalising, $\alpha \mid\!\sim \beta$. Then we get, using *Cut*, that $\alpha \mid\!\sim \gamma$ and again, since $\leadsto$ is internalising, $\mid\!\sim (\alpha \leadsto \gamma)$. Thus $\overline{\alpha \leadsto \beta} \leq \overline{\alpha \leadsto \gamma}$.

Now, by the hypothesis, $\mid\!\sim (\alpha \leadsto_s \beta$ implies $\alpha \mid\!\sim \beta$ and thus, since $\leadsto$ is internalising, $\mid\!\sim (\alpha \leadsto \beta)$. This means $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \beta}$. By transitivity we have $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}$.
We have thus proved that, if $\overline{\alpha} \wedge \overline{\beta} \leq \overline{\gamma}$, then $\mid\!\sim \alpha \leadsto_s \beta$ implies $\mid\!\sim \alpha \leadsto \gamma$ for any $\mid\!\sim \in \mathcal{C}$, which means $\overline{\alpha \leadsto_s \beta} \leq \overline{\alpha \leadsto \gamma}$. We now get by (2), (3) and Mittelstaedt's Theorem (see Chapter 3) that $\langle \overline{Fml}, \leq, ^* \rangle$ and thus $\langle Prop, \subset, ^* \rangle$ are orthomodular and, moreover, $\overline{\alpha \leadsto \beta} = \overline{\alpha}^* \vee (\overline{\alpha} \wedge \overline{\beta})$. From this it follows that $\leadsto$ and $\leadsto_s$ are $\mathcal{C}$-equivalent.
That the lattices under consideration are Boolean if $\leadsto_s$ is $\mathcal{C}$-equivalent to material implication, again, follows by Mittelstaedt's theorem.. This completes the proof. ▮

Remark: It should be pointed out that in the above proof two 'logical' properties of the class $\mathcal{C}$ play a crucial role in establishing the fact that the lattices $\langle \overline{Fml}, \leq, ^* \rangle$ and $\langle Prop, \subset\ ^* \rangle$ have the algebraic property of being orthomodular. The first 'logical' property is that an internalising connective having a certain property exists for the action $\langle \mathcal{C}, F \rangle$. This property of an action can, as we shall see, be viewed as a generalisation of the property that the Deduction Theorem holds. The second crucial property is that all consequence relations of $\mathcal{C}$ satisfy *Cut*.
For the purposes of this paper we introduce the following notion of a *logic*.

Definition 9.4 *Let $\langle \mathcal{C}, F \rangle$ be a CRS and $\leadsto$ an internalising connective for $\langle \mathcal{C}, F \rangle$. Then call the triple $\mathcal{L} = \langle \mathcal{C}, F, \leadsto \rangle$ a logic.*

We may thus interpret the above theorem as essentially saying that for a CRS to become a logic (with $\leadsto_s$ as its internalising connective), it is necessary that the lattice of operators $\langle \overline{Fml}, \leq, ^* \rangle$ and thus the lattice of propositions $\langle Prop, \subset, ^* \rangle$ have the algebraic property of being orthomodular.
Given a consequence relation $\mid\!\sim$, then define $C(\mid\!\sim) =: \{\alpha \mid \mid\!\sim \alpha\}$. We have the

Proposition 9.1 *Let $\mathcal{L} = \langle \mathcal{C}, F, \leadsto \rangle$ be a logic. Given $\mid\!\sim_1, \mid\!\sim_2 \in \mathcal{C}$. Then $C(\mid\!\sim_1) = C(\mid\!\sim_2)$ iff $\mid\!\sim_1 = \mid\!\sim_2$.*

Proof. Suppose $\mathcal{C}(\mid\!\sim_1) = \mathcal{C}(\mid\!\sim_2)$ and let $\alpha \mid\!\sim_1 \beta$. It follows, since $\leadsto$ is internalising that $\mid\!\sim_1 (\alpha \leadsto \beta)$ and thus by the hypothesis $\mid\!\sim_2 (\alpha \leadsto \beta)$. Again, since $\leadsto$ is internalising, we get $\alpha \mid\!\sim_2 \beta$, thus $\mid\!\sim_1 \subset \mid\!\sim_2$. By symmetry we also get the other inclusion. ▮

9.1.5 Classical Logic Revisited

Let us now return to our motivating example from classical logic and look at it from the point of view of the framework developed in the last subsection. Let $\vdash$ denote classical consequence and let $\Sigma \subset Fml$ be any consistent set of formulas. Define the class $\mathcal{C}_{\Sigma,\alpha}$ of consequence relations as follows. For a given formula α, define $\vdash_{\Sigma,\alpha}$ by:

$$\beta \vdash_{\Sigma,\alpha} \gamma \text{ iff } \Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma$$

Moreover, define $\mathcal{C}_{\Sigma,L} = \{\vdash_{\Sigma,\alpha} \mid \alpha \in Fml\}$ and the function $\mathcal{F}_{\Sigma,L} : Fml \times \mathcal{C}_{\Sigma,L} \to \mathcal{C}_{\Sigma,L}$ by $\mathcal{F}_{\Sigma,L}(\alpha, \vdash_{\Sigma,\alpha}) = \vdash_{\Sigma,\alpha\wedge\beta}$. It is immediately verified, using familiar facts of classical logic such as the Deduction Theorem, that consequence relations as defined above satisfy all the conditions we imposed and that $\langle \mathcal{C}_{\Sigma,L}, \mathcal{F}_{\Sigma,L} \rangle$ is a CRS. We have

$$\vdash_{\Sigma,\alpha} = \vdash_{\Sigma,\beta} \text{ iff } \Sigma \vdash \alpha \leftrightarrow \beta$$

Theorem 9.3 *$\mathcal{L}_{L,\Sigma} = \langle \mathcal{C}_{L,\Sigma}, \mathcal{F}_{L,\Sigma}, \to \rangle$ is a logic. The lattice of operators $\mathcal{O}_{\mathcal{L}_{L,\Sigma}}$ and thus the lattice of propositions $\mathcal{P}_{\mathcal{L}_{L,\Sigma}}$ are Boolean algebras isomorphic to the Lindenbaum-Tarski algebra $\mathcal{B}(\Sigma)$.*

Proof. For the first part of our claim we need to prove that $\to$ is an internalising connective for $\langle \mathcal{C}_{L,\Sigma}, \mathcal{F}_{L,\Sigma} \rangle$. But this is exactly what the Deduction Theorem says:

$$\Sigma \cup \{\alpha\} \vdash (\beta \to \gamma) \text{ iff } \Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma$$

It follows from the fact that $\to$ is internalising and Theorem ? that the lattices under consideration are Boolean algebras. Moreover, it is straightforward to prove that the following function $\varphi : \mathcal{O}_{\mathcal{L}_{L,\Sigma}} \to \mathcal{B}(\Sigma)$ is well defined and is an isomorphism

$$\varphi(\overline{\alpha}) = \alpha_\Sigma,$$

where α_Σ denotes the (unique) element of the Lindenbaum-Tarski algebra $\mathcal{B}(\Sigma)$ to which α belongs. ▮

It is interesting to note that we have established the well known fact that the Lindenbaum algebra is Boolean (see Chapter 2), in a way, however, which is radically different from the method usually applied.

Note that in case that Σ is a complete theory all the algebras we consider, namely $\mathcal{O}_{\mathcal{L}_{L,\Sigma}}$, $\mathcal{P}_{L,\Sigma}$ and the Lindenbaum-Tarski algebra $\mathcal{B}(\Sigma)$ are trivial Boolean algebras, i.e. consisting of 0 and 1 only.
In view of Theorem ? we have the follwing

Proposition 9.2 *Let $\mathcal{B}$ be any Boolean algebra. Then there exists a class of consequence relations $\mathcal{C}$ and a function F such that $\mathcal{L} = \langle \mathcal{C}, F, \to \rangle$ is a logic with material implication as its internalising connective and $\mathcal{B}$ is isomorphic to its operator algebra $\mathcal{O}_{\mathcal{L}}$.*

The above fact gives rise to the following question:
Is it true that for every orthomodular lattice $\mathcal{O}$ there exists a logic $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow_s \rangle$ with the Sasaki hook as its internalising connective such that $\mathcal{O}$ is isomorphic to $\mathcal{O}_{\mathcal{L}}$?

9.1.6 The Semantics of Consequence Revision Systems

The Concept of an $\mathcal{H}$-Model

In most of the traditional approaches to logic, a logic can, syntactically, be viewed as a class of formulas and, semantically, the corresponding class of models is a *class of models for formulas.* In our approach, the analogue of the class of formulas in the traditional approaches is given by *a class of consequence relations.* It would, therefore, be natural that the class of models should be *a class of models for consequence relations* rather than a class of models for formulas. For quite some time in the history of logic it was not clear what such a thing could be. As already mentioned, however, such a type of model was put forward in the seminal paper by Kraus-Lehmann-Magidor [34] . We shall use KLM as an abbreviation for these three names. The models we use in this paper are $GKLM$ (Generalised Kraus–Lehmann–Magidor) models as defined in Chapter 2.

We now propose the following concept of a model for a CRS.

Definition 9.5 *Let $\langle \mathcal{C}, F\rangle$ be a CRS and $\mathcal{H} = \langle H, h, \mathcal{F}, l, g\rangle$ a structure such that*

- *H is a non-empty set*
- *$h : H \rightarrow \mathcal{C}$ is a surjective function*
- *$\mathcal{F} : Fml \times H \rightarrow H$ is a function inducing F on $Fml \times \mathcal{C}$ via h, i.e. $F(\alpha, h(x)) = h(\mathcal{F}(\alpha, x))$*
- *l is a function assigning to every $x \in H$ a set of Scott-models.*
- *g is an injective function assigning to every $x \in H$ a binary relation $\leq_x \subset H \times H$ such that $\mathcal{M}_x = \langle H, \leq_x, l\rangle$ is a $GKLM$ model for $h(x)$.*

Then we say that $\mathcal{H}$ is an $\mathcal{H}$-model for $\langle \mathcal{C}, F\rangle$. We say that $\mathcal{H}$ is an $\mathcal{H}$-model for the logic $\langle \mathcal{C}, F, \rightsquigarrow\rangle$ if $\mathcal{H}$ is an $\mathcal{H}$-model for $\langle \mathcal{C}, F\rangle$. For $x \in H$ and $\alpha \in Fml$ define

$$\langle \mathcal{H}, x\rangle \models \alpha \text{ iff } s(\alpha) = 1 \text{ for all } s \in l(x).$$

We say : α is true at x in $\mathcal{H}$.

Remark: Note that we say that a formula is true at x iff it is provable in $h(x)$. The following Propositions serve to illustrate the nature of $\mathcal{H}$-models. The proofs are obvious from the definition of an $\mathcal{H}$-model.

Proposition 9.3 *Let $\langle \mathcal{C}, F\rangle$ be a CRS and $\mathcal{H}$ be an $\mathcal{H}$-model for $\mathcal{C}, F\rangle$. Let $\mid\sim \in \mathcal{C}$ and $x \in H$ such that $h(x) = \mid\sim$. Then the following conditions are equivalent.*

$$\textit{(i)}\ \alpha \mid\sim \beta$$

$$\textit{(ii)}\ \mathcal{M}_x \models \alpha \mid\sim \beta$$

$$\textit{(iii)}\ \mathcal{M}_{\mathcal{F}(\alpha,x)} \models\mid\!\sim \beta$$

$$\textit{(iv)}\ (\mathcal{H}, \mathcal{F}(\alpha, x)) \models \beta$$

$$\textit{(v)}\ \mid\!\sim_\alpha \beta$$

Proposition 9.4 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a logic and $\mathcal{H}$ an $\mathcal{H}$-model for $\langle \mathcal{C}, F \rangle$. Let $x \in H$. Then the following conditions are equivalent*

$$\textit{(i)}\ \alpha \vdash_{\mathcal{M}_x} (\beta \rightsquigarrow \gamma)$$

$$\textit{(ii)}\ \langle \mathcal{H}, \mathcal{F}(\alpha, x) \rangle \models \beta \rightsquigarrow \gamma$$

$$\textit{(iii)}\ \langle \mathcal{H}, x \rangle \models \alpha \rightsquigarrow (\beta \rightsquigarrow \gamma)$$

$$\textit{(iv)}\ \beta \vdash_{\mathcal{M}_{\mathcal{F}(\alpha,x)}} \gamma$$

$$\textit{(v)}\ \beta \mid\!\sim \gamma, \textit{ where } \mid\!\sim = F(\alpha, h(x)).$$

$$\textit{(v)}\ \alpha \mid\!\sim (\beta \rightsquigarrow \gamma) \textit{ with } \mid\!\sim = h(x)$$

The Fibred Mode of Evaluation in $\mathcal{H}$-Models

The notion of an $\mathcal{H}$-model serves a double purpose. First, it makes sense to speak of the *truth* of a formula in such a model as we are used to from traditional logics and their model theory.
Second, these models reflect the following feature of our logics. To see this, recall what intuitively the function of an internalising connective is. An internalising connective serves to reflect the metaconcept of consequence at the object level. So, intuitively, formulas containing the internalising connective 'talk' about consequence. $\mathcal{H}$-models account for this in that they not only model the truth of such formulas but also explicitly model the statements about consequence these formulas make. This means that in the process of evaluation of a formula in an $\mathcal{H}$-model the internalising connective is evaluated in a $GKLM$ model.
Given an $\mathcal{H}$-model $\mathcal{H}$, $x \in H$ and a formula of the form $\alpha \rightsquigarrow \beta$. We then have two ways of evaluating the internalising connective $\rightsquigarrow$. The first way of doing this is to check whether $\langle \mathcal{H}, x \rangle) \models (\alpha \rightsquigarrow \beta)$ according to the definition of truth given above. The second way of evaluating the connective $\rightsquigarrow$ is to look at the $GKLM$ model $\mathcal{M}_x$ and check whether $\alpha \vdash_{\mathcal{M}_x} \beta$. If so, we have, since $\mathcal{M}_x$ is a $GKLM$ model for $h(x) =: \mid\!\sim$, $\alpha \mid\!\sim \beta$. We have $\alpha \vdash_{\mathcal{M}_x} \beta$ iff $\langle \mathcal{H}, x \rangle \models (\alpha \rightsquigarrow \beta)$. This is how the $\mathcal{H}$-model reflects the fact that $\rightsquigarrow$ is an internalising connective.
Let us now look at a more complex formula. Consider a formula of the form

$$\varphi = (\alpha \rightsquigarrow (\beta \rightsquigarrow (\gamma \rightsquigarrow \delta)))$$

We may now proceed as follows.
We evaluate φ in the $GKLM$ model $\mathcal{M}_x$.We have

$$\alpha \vdash_{\mathcal{M}_x} (\beta \rightsquigarrow (\gamma \rightsquigarrow \delta)$$

$$\text{iff } \beta \vdash_{\mathcal{M}_{\mathcal{F}(\alpha,x)}} (\gamma \rightsquigarrow \delta)$$

$$\text{iff } \gamma \vdash_{\mathcal{M}_{\mathcal{F}(\beta,(\mathcal{F}(\alpha,x))}} \delta$$

We have

$$\langle \mathcal{H}, x \rangle \models \varphi \text{ iff } \gamma \vdash_{\mathcal{M}_{\mathcal{F}(\beta,\mathcal{F}(\alpha,x))}} \delta$$

The characteristic feature of the second mode of evaluation is that the connective $\rightsquigarrow$ is evaluated in $GKLM$ models as consequence. At each stage in the process of evaluation we have to switch from one $GKLM$ model to another using the 'fibring function'

$$\mathcal{F}^* : Fml \times \mathcal{M} \to \mathcal{M}, \text{ where } \mathcal{M} =: \{\mathcal{M}_x \mid x \in H\}$$

defined by

$$\mathcal{F}^*(\alpha, \mathcal{M}_x) = \mathcal{M}_{\mathcal{F}(\alpha,x)}$$

Note that this 'fibring function' is well defined since by the last clause of the definition of an $\mathcal{H}$-model we have $\mathcal{M}_x = \mathcal{M}_{x'}$ iff $x = x'$. The mode of evaluation just presented is in the spirit of what was put forward by Gabbay in several papers and his books [18] and [**?**]. as fibred semantics.

9.1.7 $\mathcal{H}$-Models and Classical Logic

We shall now show that the concept of an $\mathcal{H}$-model arises in a natural way in connection with the logics $\mathcal{L}_{L,\Sigma} = \langle \mathcal{C}_{L,\Sigma}, \mathcal{F}_{L,\Sigma}, \to \rangle$, i.e. classical logic. In Chapyer 8 we will see how this concept occurs naturally in connection with the logics arising in connection with Hilbert spaces.

Definition 9.6 *Let Σ be a set of formulas consistent in classical logic. Consider the structure $\mathcal{H}_{L,\Sigma} =: \langle \mathcal{C}_{L,\Sigma}, h, \mathcal{F}_{L,\Sigma}, l_\Psi, g \rangle$ such that*

- *h is the identity function.*
- *The function l is defined as follow. $l(\vdash_{\Sigma,\alpha}) = \{s_\alpha\}$, where $s_\alpha(\beta) = 1$ iff $\vdash_{\Sigma,\alpha} \beta$, else 0.*
- *The function g is defined as follows. Given $x = \vdash_{\Sigma,\alpha} \in \mathcal{C}_{L,\Sigma}$, then define $g(x) =: \leq_\alpha$ as follows: $\vdash_{\Sigma,\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$ is defined only if $\vdash_{\Sigma,\beta} \alpha$. Then, if $\vdash_{\Sigma,\gamma} \alpha$, $\vdash_{\Sigma,\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$ iff $\vdash_{\Sigma,\gamma} \beta$. If not $\vdash_{\Sigma,\gamma} \alpha$, then $\vdash_{\Sigma,\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$*

Note that in the above definition the function l and g are well defined. This is readily seen using familiar facts of classical logic.

Note that we use the notation $[\alpha]$ in two different contexts, namely in the context of a CRS and in the context of a $GKLM$ model. In the present situation the notions coincide, since we have $\vdash_{\Sigma,\alpha} \beta$ iff $s_\alpha(\beta) = 1$

Lemma 9.9 *If $\vdash_{\Sigma,\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$ and $\vdash_{\Sigma,\gamma} \leq_\alpha \vdash_{\Sigma,\beta}$, then $\vdash_{\Sigma,\beta} = \vdash_{\Sigma,\gamma}$.*

Proof. Assume that $\vdash_{\Sigma,\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$ and $\vdash_{\Sigma,\gamma} \leq_\alpha \vdash_{\Sigma,\beta}$. Then we observe, inspecting the definition of $\leq_\alpha$, that we have both $\vdash_{\Sigma,\beta} \alpha$ and $\vdash_{\Sigma,\gamma} \alpha$. But in this case, again by the definition of $\leq_\alpha$, the above is only possible if $\vdash_{\Sigma,\beta} \gamma$ and $\vdash_{\Sigma,\gamma} \beta$. From this it follows by classical logic that $\vdash_{\Sigma,\beta} = \vdash_{\Sigma,\gamma}$. ▮

Lemma 9.10 *$\vdash_{\Sigma,\alpha\wedge\beta}$ is the unique $\leq_\alpha$-minimal element in $[\beta]$, where $[\beta]$ denotes the proposition represented by β in the logic $\mathcal{L}_{L,\Sigma}$.*

Proof. First note that $\vdash_{\Sigma,\alpha\wedge\beta} \in [\beta]$, since $\vdash_{\Sigma,\alpha\wedge\beta} \beta$. Clearly, $\vdash_{\Sigma,\alpha\wedge\beta} \alpha$. Let $\vdash_{\Sigma,\gamma} \in [\beta]$. If not $\vdash_{\Sigma,\gamma} \alpha$ then $\vdash_{\Sigma,\alpha\wedge\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$. If $\vdash_{\Sigma,\gamma} \alpha$ then, since $\vdash_{\Sigma,\gamma} \beta$, $\vdash_{\Sigma,\gamma} \alpha \wedge \beta$. But this means $\vdash_{\Sigma,\alpha\wedge\beta} \leq_\alpha \vdash_{\Sigma,\gamma}$ for every $\vdash_{\Sigma,\gamma} \in [\beta]$. From this and the last lemma it follows that $\vdash_{\Sigma,\alpha\wedge\beta}$ a $\leq_\alpha$-minimal element in $[\beta]$. To see that it is unique, let $\vdash_{\Sigma,\delta}$ be any $\leq_\alpha$-minimal element of $[\beta]$. We have $\vdash_{\Sigma,\alpha\wedge\beta} \leq_\alpha \vdash_{\Sigma,\delta}$. Since $\vdash_{\Sigma,\delta}$ is $\leq_\alpha$-minimal in $[\beta]$ we get $\vdash_{\Sigma,\delta} = \vdash_{\Sigma,\alpha\wedge\beta}$. ▮

Theorem 9.4 *$\mathcal{H}_{L,\Sigma}$ is an $\mathcal{H}$-model for $\mathcal{L}_{L,\Sigma}$.*

Proof. We need to prove that for every $x = \vdash_{\Sigma,\alpha} \in \mathcal{C}_{\Sigma,L}$, $\mathcal{M}_x = \langle \mathcal{C}_{\Sigma,L}, \leq_x, l \rangle$ is a $GKLM$ model for x. We have smoothness by Lemma 17

Suppose $\beta \vdash_{\Sigma,\alpha} \gamma$. By definition this means $\Sigma \cup \{\alpha \wedge \beta\} \vdash \gamma$, which is equivalent to $\vdash_{\Sigma,\alpha\wedge\beta} \in [\gamma]$ and our claim follows from Lemma ? and Definition ?. ▮

9.2 The Concept of a Holistic Logic

We now have the technical equipment to present the logical structure that correspond to the intuitions discussed earlier in this chapter.

Let us start from our motivating example. In that example the consequence relations cannot be 'characterised' by a single formula, i.e. given any $\vdash_\alpha$, then there exists no formula β such that β is provable in $\vdash_\alpha$ and only in $\vdash_\alpha$. We take this observation as a motivation for studying logics in which every consequence relation has a 'characterising' formula.

Definition 9.7 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a logic with the following properties.*

- *For any non-zero $\mid\!\sim_0 \in \mathcal{C}$ there exists a formula $\sigma_{\mid\sim_0}$ such that $\mid\!\sim \sigma_{\mid\sim_0}$ iff $\mid\!\sim = \mid\!\sim_0$. We call σ a pointer to $\mid\!\sim$.*
- *For every $\mid\!\sim \in \mathcal{C}$ there exist a formula α such that neither $\mid\!\sim \alpha$ nor $\mid\!\sim \neg\alpha$.*

We call $\mathcal{L}$ a (non-degenerate) holistic logic if both of the above conditions are satisfied. We call $\mathcal{L}$ degenerate holistic if the first condition is satisfied bur not necessarily the second one. We call $\mathcal{L}$ totally degenerate if no consequence relation satisfies the second condition.

Remarks: Any two pointers σ_1 and σ_2 to the same consequence relation are (globally) equivalent, i.e. $[\sigma_1] = [\sigma_2]$ We assume the consequence relation referred to later to be consistent without expplicit mentioning.

Intuitively, the second condition says that every consequence relation $\mid\!\sim$ must be genuinely revisable, i.e. we assume that there exiats a formula α such that $\mid\!\sim_\alpha$ is consistent and distinct from $\mid\!\sim$. Assume we have consequence relations in a *CRS* that are not genuinely revisable. Then we can 'take them out' of the revision system and restrict 'proper' revision to the rest so as to get a non-degenerate system. It follows that a (non degenerate) holistic logic has at least two consequence relations. We will always use the term 'holistic' in the sense of 'non degenerate holistic' except in the theorem which we call the Limiting Case Theorem. In the case of a totally degenerate holistic logic there is no genuine revision at all.

In the next subsections we state some salient properties of holistic logics.

9.2.1 Orthogonality, Encodedness, Dimension

Definition 9.8 *Let $\mathcal{L}$ be a holistic logic and let $\mid\!\sim_1$ and $\mid\!\sim_2$ be two consequence relations $\mathcal{L}$ with pointers σ_1 and σ_2 respectively. Then we say that $\mid\!\sim_1$ and $\mid\!\sim_2$ are orthogonal if $\mid\!\sim_1 \neg\sigma_2$ and $\mid\!\sim_2 \neg\sigma_1$.*

Actually it suffices to require one of the two conditions. It can then be proved, using the global condition ?, that the relation thus defined is symmetric

The following Lemma follows from the definition of a pointer and that of a consequence revision systems.

Lemma 9.11 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic, $\mid\!\sim_1, \mid\!\sim_2 \in \mathcal{C}$. If $\mid\!\sim_1$ and $_2$ are not orthogonal, then ${\mid\!\sim_1}_{\sigma_2} = \mid\!\sim_2$ and vice versa. If they are orthogonal we have ${\mid\!\sim_1}_{\sigma_2} = 0$ and vice versa.*

Lemma 9.12 *Let $\mathcal{L}$ be a holistic logic and $\mid\!\sim_1$ and $\mid\!\sim_2$ two non-orthogonal consequence relations. Then we have $\alpha \mid\!\sim_1 \beta$ iff $\sigma_1 \mid\!\sim_2 (\alpha \rightsquigarrow \beta)$ and vice versa.*

Proof. Recall that $\mid\!\sim_1$ and $\mid\!\sim_2$ are non-orthogonal iff ${\mid\!\sim_2}_{\sigma_1} = \mid\!\sim_1$ and vice versa. We have $\alpha \mid\!\sim_2 \beta$ iff $\mid\!\sim_2 \alpha \rightsquigarrow \beta$, since $\rightsquigarrow$ is internalising. $\alpha \mid\!\sim_2 \beta$ is thus equivalent to ${\mid\!\sim_2}_{\sigma_1} \alpha \rightsquigarrow \beta$. But this is by Lemmas ? and ? the case iff $\sigma_1 \mid\!\sim_2 \alpha \rightsquigarrow \beta$.

∎

The above lemma says that non-orthogonal consequence relations are 'encoded' in each other. This fact is the motivation for calling these structures 'holistic'.

Definition 9.9 *Let $\mathcal{L}$ be a holistic logic. We call a family ${\mid\!\sim}_{i(i \in I)}$ of pairwise orthogonal consequence relations a basis of $\mathcal{L}$ if for any $\mid\!\sim$ of $\mathcal{L}$ there exists a basis consequence relation $\mid\!\sim_j$ not orthogonal to $\mid\!\sim$. We call $\mathcal{L}$ finite- dimensional iff it admits a finite basis. If $\mathcal{L}$ is finite-dimensional we say it has dimension n if it admits a basis of n elements and no basis of fewer elements.*

Remark: We may view the basis consequence relations $\mid\!\sim_i$ as containing all the 'information' of the logic $\mathcal{L}$ in the sense that every consequence relation is encoded in at least one basis consequence relation.

Lemma 9.13 *Let $\mathcal{L} = \langle \mathcal{C}, \mathcal{F}, \rightsquigarrow \rangle$ be a holistic logic and $\mid\!\sim \in \mathcal{C}$ with pointer σ. Then we have*

- *(i) $\mid\!\sim \alpha$ iff $[\sigma \rightsquigarrow \alpha] = [\top]$ and thus $[\neg(\sigma \rightsquigarrow \alpha)] = [\bot]$*
- *(ii) $\not\mid\!\sim \alpha$ iff $[\sigma \rightsquigarrow \alpha] = [\neg\sigma]$ and thus $[\neg(\sigma \rightsquigarrow \alpha)] = [\sigma]$*

Proof. (i) For the direction from left to right suppose $\mid\!\sim \alpha$ and note that for any $\mid\!\sim_1$ orthogonal to $\mid\!\sim_1$ we have $\mid\!\sim_1 \sigma \rightsquigarrow \alpha$. If $\not\mid\!\sim_1$ non-orthogonal to $\mid\!\sim$ we have by 'encodedness' that $\mid\!\sim \sigma \rightsquigarrow \alpha$. Thus $[\sigma \rightsquigarrow \alpha] = \mathcal{C} = [\top]$. The other direction is obvious.
(ii) Suppose that $\not\mid\!\sim \alpha$. Then, again, we have for every $\mid\!\sim_1$ orthogonal to $\mid\!\sim$ that $\mid\!\sim_1 \sigma \rightsquigarrow \alpha$. But if $\mid\!\sim_1$ is not orthogonal to $\mid\!\sim$, $\mid\!\sim_1 \sigma \rightsquigarrow \alpha$ cannot hold, since this would imply $\mid\!\sim \alpha$ contrary to the hypothesis. Thus $[\sigma \rightsquigarrow \alpha] = [\neg\sigma]$. The other direction is obvious. ▮

Remark: Thus the propositions corresponding to formulas which are translations of metastatements have the form $[\sigma]$, $[\neg\sigma]$, $[\top]$, $[\bot]$. They form a Boolean algebra in a natural way.

9.2.2 Selfreferential Soundness and Completeness

In this section we introduce the notion of self-referential completeness in connection with consequence relations. This notion was first introduced by Smullyan in [59] for modal systems. We shall prove that the consequence relations of a holistic logic are self-referentially sound and complete and, with a certain trivial exception, non-monotonic.

We now define a meta-language in which we can talk about provability. Intuitively, $DER(\alpha, \beta)$ means "β is derivable from α in $\mid\!\sim$".

Definition 9.10
- *(i) If α, β are wffs of the object language, then $DER(\alpha, \beta) \in ML$.*
- *If α is a wff of the object language and $\varphi \in ML$, then $DER(\alpha, \varphi) \in ML$ and $DER(\varphi, \alpha) \in ML$.*
- *If $\varphi, \psi \in ML$, then $DER(\varphi, \psi) \in ML$.*
- *If $\varphi, \psi \in ML$, so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$.*

We use the following abbreviations:

$$PROV\alpha =: DER(\top, \alpha)$$

$$CON\alpha =: \neg PROV\neg\alpha$$

$$EQUIV(\alpha, \beta) =: DER(\alpha, \beta) \wedge DER(\beta, \alpha)$$

We now define a natural translation of the meta-language ML into the object language. We assume that we have a logic $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$. The following definitions are relative to a fixed $\mid\!\sim \in \mathcal{C}$ having a pointer σ to itself.

Definition 9.11 *Let σ be a pointer to $\mid\!\sim$. Define the translation $'$ as follows.*

- *(i) If $\varphi = DER(\alpha, \beta)$ where α and β are formulas of the object language, $\varphi' = \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$*
- *(ii) If $\varphi = DER(\alpha, \psi)$, where α is a formula of the object language and $\psi \in ML$, then $\varphi' = \sigma \rightsquigarrow (\alpha \rightsquigarrow \psi')$; analogously for the case $DER(\psi, \alpha)$*
- *(iii) If $\varphi = DER(\psi, \rho)$, where $\psi, \rho \in ML$ $\varphi' = \sigma \rightsquigarrow (\psi' \rightsquigarrow \rho')$*
- *(iv) If $\varphi = \neg\psi$, $\varphi' = \neg(\sigma \rightsquigarrow \psi')$,*
- *(v) If $\varphi = \psi \wedge \rho$, $\varphi' = \psi' \wedge \rho'$; analogously for the other connectives.*

We now define the notion of truth for ML in a natural way. This definition of truth is in the spirit of what Smullyan calls a self-referential interpretation in the above mentioned books.

Just recall the essential feature of the definition of self-referential truth. Given a modal system M. Then we say that a formula of the form $\Box A$ is (self-referentially) true with respect to M iff A is provable in M.

Definition 9.12

- *(i) If $\varphi = DER(\alpha, \beta)$, where α, β are formulas of the object language, then $TRUE\ \varphi$ iff $\alpha \mid\!\sim \beta$*
- *(ii) If $\varphi = DER(\alpha, \psi)$, where α is a wff of the object language, then $TRUE\ \varphi$ iff $\alpha \mid\!\sim \psi'$; analogously for the case $DER(\psi, \alpha)$*
- *(iii) If $\varphi = DER(\psi, \rho)$ for $\psi, \rho \in ML$, then $TRUE\ \varphi$ iff $\psi' \mid\!\sim \rho'$.*
- *(iv) If $\varphi = \neg\psi$, then $TRUE\ \varphi$ iff not $TRUE\ \psi$; analogously for the other connectives.*

Definition 9.13 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic, $\mid\!\sim \in \mathcal{C}$ and σ a pointer to $\mid\!\sim$. We say that $\mid\!\sim$ is self-referentially sound if for all $\varphi \in ML$ we have:*

$$\mid\!\sim \varphi' \text{ implies } TRUE\ \varphi.$$

We say that $\mid\!\sim$ is self-referentially complete if

$$TRUE\ \varphi \text{ implies } \mid\!\sim \varphi'$$

Lemma 9.14 *Given a non-zero $\mid\!\sim_0, \mid\!\sim \in \mathcal{C}$, with pointer σ such that $\not\mid\!\sim \neg\sigma$. Then*

$$\text{(i) } \alpha \mid\!\sim_0 \beta \text{ iff } \mid\!\sim \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$$

Moreover, we have

- *(ii)* $\mathrel{|\!\sim} \alpha$ *iff* $[\sigma \rightsquigarrow \alpha] = [\top]$ *and thus* $[\neg(\sigma \rightsquigarrow \alpha)] = [\bot]$
- *(iii)* $\not\mathrel{|\!\sim} \alpha$ *iff* $[\sigma \rightsquigarrow \alpha] = [\neg\sigma]$ *and thus* $[\neg(\sigma \rightsquigarrow \alpha)] = [\sigma]$

Remark: We may view the formula $\neg(\sigma \rightsquigarrow \alpha)$ as expressing the unprovability of α at the object level. It thus follows from (iii) of the above lemma that, if α is not provable, then its unprovabilty can be proved. This is a remarkable fact unfamiliar from classical logic which has far-reaching consequences concerning self-referential completeness and non-monotonicity.

Theorem 9.5 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic, $\mathrel{|\!\sim} \in \mathcal{C}$ with pointer σ.Then $\mathrel{|\!\sim}$ is self-referentially sound and complete.*

Proof. By induction on the construction of the formulas of ML.
(i) Case $\varphi = DER(\alpha, \beta)$. By definition $TRUE\ \varphi$ means $\alpha \mathrel{|\!\sim} \beta$. But this means $\mathrel{|\!\sim} \alpha \rightsquigarrow \beta$, which is equivalent to $\mathrel{|\!\sim} \sigma \rightsquigarrow (\alpha \rightsquigarrow \beta)$. But this says that $\mathrel{|\!\sim} \varphi'$.
(ii) Case $\varphi = DER(\alpha, \psi)$. Suppose $TRUE\ \varphi$. By definition this says $\alpha \mathrel{|\!\sim} \psi'$ or equivalently $\mathrel{|\!\sim} \sigma \rightsquigarrow (\alpha \rightsquigarrow \psi')$. But this is exactly what $\mathrel{|\!\sim} \varphi'$ means.
(iii) Case $\varphi = DER(\psi, \rho)$. The proof is analogous to (ii).
(iv) Case $\varphi = \neg\psi$. $TRUE\ \varphi$ means that not $TRUE\psi$. By the induction hypothesis this is equivalent to not $\mathrel{|\!\sim} \psi'$, which by 'provability of unprovability' says that $\mathrel{|\!\sim} \neg(\sigma \rightsquigarrow \psi')$. But this means $\mathrel{|\!\sim} \varphi'$.
(v) Case $\varphi = \psi \vee \rho$. First note that $\varphi' = \psi' \vee \rho'$. Suppose $TRUE\ \varphi$. It follows that $TRUE\ \psi$ or $TRUE\ \rho$. Without loss of generality assume $TRUE\ \psi$. By the induction hypothesis we have $\mathrel{|\!\sim} \psi'$ and thus $\mathrel{|\!\sim} \psi' \vee \rho'$. But this says $\mathrel{|\!\sim} \varphi'$.

For the other direction suppose $\mathrel{|\!\sim} \varphi'$. We need to prove that $TRUE\ \varphi$. There is a problem here, namely that, generally, $\mathrel{|\!\sim} \psi' \vee \rho'$ does not imply that $\mathrel{|\!\sim} \psi'$ or $\mathrel{|\!\sim} \rho'$. To overcome this obstacle we first observe by inspecting the definition of the translation that any formula occurring as a translation is of the form $\sigma \rightsquigarrow ...$ or $\neg(\sigma \rightsquigarrow ...$ or a Boolean combination of such formulas. It then follows by Lemma **??** that the propositions $[\psi']$ and $[\rho']$ are of the form $[\top]$, $[\bot]$, $[\sigma]$, $[\neg\sigma]$. We can thus treat this case by checking all combinations.

Suppose for instance that $[\psi'] = [\top]$ and $[\rho'] = [\neg\sigma]$. Then $\mathrel{|\!\sim} \psi' \vee \rho'$ says $\mathrel{|\!\sim} \top \vee \neg\sigma$, which is equivalent to $\mathrel{|\!\sim} \top$, i.e. $\mathrel{|\!\sim} \psi'$. It follows by the induction hypothesis that $TRUE\ \psi$ and thus $TRUE\ \varphi$.

The other combinations can be checked in an analogous manner. The same applies to the other cases. ∎

Remark: Inspecting the translation of the metalanguage into the object language, we may view the metalanguage as a 'sublanguage' of the object language. The peculiar feature of this 'sublanguage' is that it contains a 'proof operator', namely $\sigma \rightsquigarrow$, as opposed to 'proof predicates' which we have in other languages. Our notion of self-referentiality thus becomes fully analogous to that introduced by Smullyan in connection with self-application of modal systems, where the modal operator $\Box$ plays the role of a proof operator.
Example: Let us consider an example and and let us for the sake of illustration verify the truth of the claim made in the above theorem directly. Let α be an object formula and consider the following metastatement

$$\varphi = PROV\alpha \to CON\alpha$$

Its translation is

$$\varphi' = (\sigma \rightsquigarrow (\top \rightsquigarrow \alpha)) \to \neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha)))$$

Let us first verify that $TRUE\ \varphi$ implies $\mid\!\sim \varphi'$. Assume that not $TRUE\ PROV\alpha$. This means $TRUE \neg PROV\alpha$, which says that $\not\mid\!\sim \alpha$. By Lemma **??** we have $[\neg(\sigma \rightsquigarrow (\top \rightsquigarrow \alpha))] = [\sigma]$. Thus $[\varphi'] = [\sigma \vee ...]$ and we have $\mid\!\sim \varphi'$.
Now assume $TRUE\ CON\alpha$, i.e. $\not\mid\!\sim \neg\alpha$ and thus $\not\mid\!\sim \top \rightsquigarrow \neg\alpha$, hence $\not\mid\!\sim \sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha)$. In this case we have by Lemma 20 $[\neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \neg\alpha)))] = [\sigma]$. Thus $[\varphi'] = [... \vee \sigma]$ and we have $\mid\!\sim \varphi'$.
Let us now verify that $\mid\!\sim \varphi'$ implies $TRUE\ \varphi$. So assume $\mid\!\sim \varphi'$. $[\neg(\sigma \rightsquigarrow (\sigma \rightsquigarrow (\top \rightsquigarrow \alpha)))]$ equals either $[\bot]$ or $[\sigma]$. In the first case we have $\mid\!\sim \alpha$. Since $\mid\!\sim$ is assumed to be consistent, we have $\not\mid\!\sim \neg\alpha$, which means $TRUE\ CON\alpha$. But this says that $TRUE\ \varphi$.
In the second case we have $\not\mid\!\sim \alpha$ and thus not $TRUE\ PROV\alpha$ in which case again $TRUE\ \varphi$.

Here are a few examples of self-referentially true and thus provable metaformulas.
We have $TRUE\varphi$ for the following metastatements.

- $\varphi = \neg EQUIV(\alpha, \neg PROV\alpha)$
- $\varphi = CON\alpha \to \neg DER(\alpha, \neg PROV\alpha)$
- $\varphi = PROV\alpha \leftrightarrow EQUIV(\alpha, \neg PROV\bot)$

By self-referential completeness we have in all three cases $\mid\!\sim \varphi'$. Note that for the first of the examples this means that the consequence relation 'knows' that it has no Gödel fixed points. We will elaborate on this later.

9.2.3 Connection with the Modal System D

The last subsection gives rise to the consideration of a certain modal system known in modal logic as the system D. More precisely, we shall be interested only in the letterless (deictic) fragment of D. This means that we are only concerned with those theorems of D containing no propositional symbols other than $\top$ and $\bot$. Such formulas are also called modal sentences.
The system D is obtained from the modal system K by adding the following axiom.

$$\Box p \to \neg\Box\neg p$$

Having the provability interpretation of the box in mind, we may view the axiom as just stating consistency.

Note that we are only interested in the letterless form of the above axiom and, more generally, in all letterless formulas which are theorems of the system D. Call this fragment MC.

We also might have chosen the following way of introducing MC. We could have confined ourselves to the letterless fragment of the language and could have stipulated the following.

(i) All letterless substitutional instances of classical tautologies are theorems of MC.

(ii) The rules of inference for MC are necessitation and modus ponens.
It can then be proved that D is a conservative extension of the system MC thus introduced.

Lemma 9.15 *For any modal sentence A we have*

- *(i)* $\vdash_{MC} A$ *or* $\vdash_{MC} \neg A$
- *(ii)* $\vdash_{MC} \Box A$ *iff* $\vdash_{MC} A$

Proof. (i) For the modal sentences $\top$ and $\bot$ the claim is obvious.
Consider the case $A = \Box B$. Assume that not $\vdash_{MC} A$. We need to prove that $\vdash_{MC} \neg A$. We have that not $\vdash_{MC} B$, since otherwise necessitation would give us $\vdash_{MC} A$, contrary to the assumption. By the induction hypothesis we have $\vdash_{MC} \neg B$ and by necessitation $\vdash_{MC} \Box\neg B$. Now, since $\vdash_{MC} \Box\neg B \rightarrow \neg\Box B$, we get by modus ponens $\vdash_{MC} \neg\Box B$, which means $\vdash_{MC} \neg A$. The case of Boolean combinations is straightforward.
(ii) Assume $\vdash_{MC} \Box A$ and not $\vdash_{MC} A$. By (i) we then have that $\vdash_{MC} \neg A$ and thus by necessitation $\vdash_{MC} \Box\neg A$. Since $\vdash_{MC} \Box\neg A \rightarrow \neg\Box A$, modus ponens gives us $\vdash_{MC} \neg\Box A$, which contradicts the consistency of the system D. ∎

As an immediate corollary of the above theorem we get, using necessitation, that not $\vdash A$ implies $\vdash \neg\Box A$. This can be viewed as the 'modal version' of 'provability of unprovability'. We could of course have established the above result semantically using the fact that the system D is complete for the class of Kripke frames such that for any possible world there exists a world accessible from it.
For the next theorem we use the terms 'self-referentially sound' and 'self-referentially complete' as defined by Smullyan in 'Forever Undecided'. Recall that the main clause in the definition of self-referential truth is

$$TRUE\ \Box A \text{ iff } \vdash_{MC} A$$

Theorem 9.6 *MC is self-referentially sound and complete. In particular it can prove its own consistency.*

Proof. For self-referential correctness it suffices to prove that the axioms are true. Let us check the axiom $\Box\top \rightarrow \neg\Box\bot$. Since $\vdash_{MC} \top$, $\Box\top$ is true and we need to show that $\neg\Box\bot$ is true. But this is the case iff the system is consistent, which is well known from modal logic.

For self-referential completeness we need to prove that for any A the truth of A implies $\vdash A$. Again, the only cases not completely obvious are the cases $A = \Box B$ and $A = \neg B$. Suppose $A = \Box B$ is true. This means $\vdash B$. By necessitation we have $\vdash \Box B$ and thus $\vdash A$. Suppose $A = \neg B$ is true. This says that B is not true and thus by the induction hypothesis not $\vdash B$. By the above lemma this means $\vdash \neg B$ and thus $\vdash A$. ∎

Since the system D is consistent, we have $TRUE\ \neg\Box\bot$ and thus by self-referential completeness $\vdash_{MC}\ \neg\Box\bot$. It is in this sense that we can say that D can prove its own consistency. Recall that the famous modal system G for instance does not have this property, because by Solovay's completeness theorem this would contradict Gödel's incompleteness theorem.

Notice at this point that the modal operator is trivial for MC.

Let us now describe the connection between the considerations of the last section and the system MC. Consider the fragment MLP of the metalanguage ML consisting of those formulas in which only $PROV$ occurs. More precisely, it is the following (meta-) language. It is the smallest set such that

- If α is a wfff of the object language, then $PROV\alpha \in MLP$.
- If $\varphi \in MLP$, then $PROV\varphi \in MLP$.
- If $\varphi, \psi \in MLP$, so are $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi$.

Let $\mid\!\sim$ be a consequence relation, α a wff of the object language. We say that α has truth value $\top$ with respect to $\mid\!\sim$, if $\mid\!\sim \alpha$, otherwise we say that α has truth value $\bot$ with respect to $\mid\!\sim$. Now, given a formula $\varphi \in MLP$ and a consequence relation $\mid\!\sim$ having a pointer to itself. Then we define φ^d to be the modal sentence resulting from φ by replacing every occurrence of $PROV$ by the modal operator $\Box$ and all occurrences of wffs of the object language by their truth values with respect to $\mid\!\sim$. Then we have the

Theorem 9.7 *Let $\mid\!\sim$ have a pointer to itself, let $\varphi \in MLP$. Then the following statements are equivalent.*

- *(i) $TRUE\ \varphi$*
- *(ii) $\mid\!\sim \varphi'$*
- *(iii) $\vdash_{MC} \varphi^d$*
- *(iv) $TRUE\ \varphi^d$*

Remark: Note that in (i) $TRUE$ is self-referential truth in the sense of theorem 13, whereas in (iv) $TRUE$ means self-referential truth in the modal sense.

Proof. The equivalence between (i) and (ii) is given by selfreferential completeness of $\mid\!\sim$, the equivalence of (iii) and (iv) is given by self-referential completeness of the modal system D.

We need to prove that (i) and (iii) are equivalent. Case $\varphi = PROV\alpha$. Assume

$TRUE\ \varphi$. This means $\mid\!\sim \alpha$ and thus $\varphi^d = \Box\top$ and, clearly, $\vdash_{MC} \varphi^d$. For the other direction note that either $\varphi^d = \Box\top$ or $\varphi = \Box\bot$. Since $\vdash_{MC} \varphi^d$ we have $\varphi^d = \Box\top$. It follows that the truth value of α must be $\top$ and thus $\mid\!\sim \alpha$. But this means $TRUE\ PROV\alpha$.
Case $\varphi = PROV\psi$. Assume $TRUE\ \varphi$. This means $\mid\!\sim \psi'$. We have $\varphi^d = \Box\psi^d$ and by the induction hypothesis $\vdash_{MC} \psi^d$. Hence $\vdash_{MC} \Box\psi^d$, which says $\vdash_{MC} \varphi^d$. For the other direction let $\vdash_{MC} \varphi^d$, i.e. $\vdash_{MC} \Box\psi^d$. By Lemma 22 this is quivalent to $\varphi \vdash_{MC} \psi^d$. The induction hypthesis yields $\mid\!\sim \psi'$. But this says $TRUE\varphi$. The other cases are straightforward. ▮

As a corollary we get a reduction theorem for $PROV$ regarded as a modality. Given any $\mid\!\sim$ of a holistic logic. Then any sequence of modalitis reduces to one containing only one $\Box$, i.e. every modality is equivalent with tespect to $\mid\!\sim$ to either $\Box$ or $\neg\Box$. Use this point of view in the formulation of the limiting case theorem! In the limit the proof operator becomes completely trivial. It is seemimgly no longer there. Talking about provability then looks like talking about the (external) world.

9.2.4 No Gödel fixed points

Proposition 9.5 *Let $\mathcal{L}, \langle \mathcal{C}, F, \rightsquigarrow\rangle$ be a holistic logic. $\mid\!\sim \in \mathcal{C}$ with pointer σ. Then $\mid\!\sim$ does not admit Gödelian fixed points, i.e. (consistent) formulas equivalent to their own unprovability. This means that there exists no (consistent) formula such that*

$$\text{(i) } \alpha \mid\!\sim \neg(\sigma \rightsquigarrow \alpha).$$

$$\text{(ii) } \neg(\sigma \rightsquigarrow \alpha) \mid\!\sim \alpha$$

Proof. Assume there exists a Gödelian fixed point α. Suppose $\mid\!\sim \alpha$. Then it follows from (i) that $\mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$. By 'provability of unprovability we would have $\not\mid\!\sim \alpha$, contrary to the hypothesis. Now suppose $\not\mid\!\sim \alpha$. Then, again by provability of unprovability $\mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$ and thus by (ii) $\mid\!\sim \alpha$, again contradicting the hypothesis. . ▮

9.2.5 Justifying logical rules

In justifying logical rules we normally invoke the notion of truth. Consider for example modus ponens:

$$\text{If } \mid\!\sim \alpha \text{ and } \mid\!\sim \alpha \rightarrow \beta, \text{ then } \mid\!\sim \beta$$

We accept this rule in classical logic because it preserves truth. If α is true and $\alpha \rightarrow \beta$ is true, then so is β. In classical logic, the concept of truth is a meta concept. So we justify logical rules at the meta level. Classical logic itself cannot justify its rules. This is different in the case of a holistic logic. Take the above example of modus ponens. In a holistic logic we do have modus ponens if we restrict the language to (translations) of metastements. The metastatement expressing modus ponens is

$$\varphi = (PROV\alpha \wedge DER(\alpha,\beta)) \to PROV\beta$$

Since φ is true, we have by self-referential completeness, $\mid\!\sim \varphi'$. It is in this sense that we may say that the logic justifies (proves) the rule of modus ponens.

9.2.6 The case of a complete classical theory

Recall the definition of $\mathcal{L}_\Sigma = \langle \mathcal{C}_\Sigma, \mathcal{F}_\Sigma, \to \rangle$ from the motivating example. We have the

Proposition 9.6 *Let Σ be a consistent set of formulas. Then $\mathcal{L}_\Sigma = \langle \mathcal{C}_\Sigma, \mathcal{F}_\Sigma, \to \rangle$ is holistic iff Σ is a complete classical theory. In this case $\mathcal{L}_\Sigma$ is (totally) degenerate. It has dimension 1 and we have $\mathcal{C} = \{\vdash_\Sigma, 0\} and F_\Sigma(\alpha, \vdash_\Sigma) = \vdash_\Sigma$ if $\alpha \in \Sigma$, else 0.*

Proof. Observe that for any α such that neither $\Sigma \vdash \neg\alpha$ nor $\Sigma \vdash \alpha$, $\vdash_\Sigma$ is a proper subset of $\vdash_\Sigma, \alpha)$. So in this case $\vdash_\Sigma$ cannot have a pointer. It follows that $\vdash_\Sigma$ can have a pointer only if for every α either $\Sigma \vdash \alpha$ or $\Sigma \vdash \neg\alpha$, i.e. Σ is a complete theory. In fact, in this case any formula α such that $\Sigma \vdash \alpha$ is a pointer to $\vdash_\Sigma$

∎

9.3 No Windows Theorems

9.3.1 The Local No Windows Theorem

Given a holistic logic $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ and $\mid\!\sim \in \mathcal{C}$ with pointer σ. Then we define $\Sigma_\sigma =: \{\alpha \mid \mid\!\sim \alpha\}$, to be the local theory of $\mid\!\sim$. We denote by Σ_g its global theory, i.e. $\Sigma_g =: \{\alpha \mid \mid\!\sim \alpha$ for all $\mid\!\sim \in \mathcal{C}\}$.

Lemma 9.16 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic and $\mid\!\sim \in \mathcal{C}$ with pointer σ. Suppose $\mid\!\sim \alpha$. Then $\sigma \rightsquigarrow \alpha \in \Sigma_g$.*

Proof. Let $\mid\!\sim_1 \in \mathcal{C}$. Suppose $\mid\!\sim_1$ is orthogonal to $\mid\!\sim$. Then $\mid\!\sim_{1)\ \sigma} = 0$ Then, clearly, $\sigma \mid\!\sim_1 \alpha$ and thus $\mid\!\sim_1 \sigma \rightsquigarrow \alpha$. Suppose $\mid\!\sim_1$ is not orthogonal to $\mid\!\sim$. In this case we have $\mid\!\sim_{1_\sigma} = \mid\!\sim$. Thus $\mid\!\sim_{1_\sigma} \alpha$. It follows that $\sigma \mid\!\sim_1 \alpha$ which means $\mid\!\sim_1 \sigma \rightsquigarrow \alpha$. We have proved that $\sigma \rightsquigarrow \alpha \in \Sigma_g$

∎

Lemma 9.17 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic and $\mid\!\sim \in \mathcal{C}$ with pointer σ. Suppose $\rightsquigarrow$ is classically equivalent to $\to$, i.e. material implication. Assume that $\Sigma_g \cup \{\sigma\}$ is classically consistent. Then we have $\mid\!\sim \alpha$ iff $\Sigma_g \cup \{\sigma\} \vdash \alpha$.*

Proof. For the direction from left to right note that $\mid\!\sim \alpha$ implies by lemma ? that $\sigma \rightsquigarrow \alpha \in \Sigma_g$ and, since $\rightsquigarrow$ is assumed to be classically equivalent to $\to$, we have $\Sigma_g \cup \{\sigma\} \vdash \alpha$. For the other direction suppose $\Sigma_g \cup \{\sigma\} \vdash \alpha$ and assume $\not\mid\!\sim \alpha$. We have $\Sigma_g \cup \{\sigma\} \vdash \sigma \to \alpha$. On the other hand we have by

'provability of unprovability' $\mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$ and thus, by the direction already proved, $\Sigma_g \cup \{\sigma\} \vdash \neg(\sigma \rightsquigarrow \alpha)$ and thus, since $\rightsquigarrow$ is classically equivalent to $\rightarrow$ $\Sigma_g \cup \{\sigma\} \vdash \neg(\sigma \rightarrow \alpha)$

$\Sigma_g \cup \{\sigma\}$ would thus be classically inconsistent contrary to the hypothesis. It follows that $\mid\!\sim \alpha$. ∎

We call the following theorem the (local) No Windows Theorem because it is reminiscent of what Leibniz in his "Monadology" says about the monads: "The monads have no windows".

Theorem 9.8 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a non-degenerate holistic logic. Suppose $\rightsquigarrow$ is classically equivalent to material implication. Let σ be any pointer. Then $\Sigma_g \cup \{\sigma\}$ is classically inconsistent. Thus, Σ_σ is classically inconsistent*

Proof. Let σ be any pointer with corresponding $\mid\!\sim \in \mathcal{C}$. Assume that $\Sigma_g \cup \{\sigma\}$ is classically consistent. Let α be such that $\mid\!\not\sim \neg\alpha$ and $\mid\!\not\sim \alpha$. By the hypothesis of non-degeneracy such a formula exists. Then we have by 'provability of unprovability' and non-monotonicity

$$(1) \mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$$

$$(2)\ \alpha \mid\!\not\sim \neg(\sigma \rightsquigarrow \alpha)$$

We have by Lemma ?

$$(3)\ \Sigma_g \cup \{\sigma\} \vdash \neg(\sigma \rightsquigarrow \alpha)$$

and thus by classical logic

$$(4)\ \Sigma_g \cup \{\sigma\} \vdash \alpha \rightarrow \neg(\sigma \rightsquigarrow \alpha)$$

Since $\rightsquigarrow$ is classically equivalent to $\rightarrow$, it follows that

$$(5)\ \Sigma_g \cup \{\sigma\} \vdash \alpha \rightsquigarrow \neg(\sigma \rightsquigarrow \alpha)$$

Again, by Lemma 27 we get

$$(6) \mid\!\sim \alpha \rightsquigarrow \neg(\sigma \rightsquigarrow \alpha)$$

and thus

$$(7)\ \alpha \mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$$

But (7) contradicts (2). It follows that $\Sigma_g \cup \{\sigma\}$ is classically inconsistent. ∎

9.3.2 The Global No Windows Theorem

We now restrict ourselves to the case of a finite-dimensional holistic logic. In this case we can sharpen the no windows theorem so as to get a Kochen-Specker type result as a special case.

Lemma 9.18 *Let $\mathcal{L}$ be any logic and α such that $\not\mid\!\sim \alpha$ for every $\mid\!\sim \neq 0$. Then $\alpha \rightsquigarrow \bot \in \Sigma_g$.*

Proof. Given any $\mid\!\sim \in \mathcal{C}$. We claim that $\mid\!\sim_\alpha = 0$. For otherwise we would have $\mid\!\sim_\alpha \alpha$ with $\mid\!\sim_\alpha \neq 0$ contrary to the hypothesis. Thus $\mid\!\sim_\alpha \bot$, which means $\alpha \mid\!\sim \bot$ and thus $\mid\!\sim \alpha \rightsquigarrow \bot$. We have proved that $\alpha \rightsquigarrow \bot \in \Sigma_g$. ∎

The following theorem is a summary of previous results and, moreover, contains the strengthened version of the no windows theorem.

Theorem 9.9 *Let $\mathcal{L} = \langle \mathcal{C}, \mathcal{F}, \rightsquigarrow \rangle$ be a non-degenerate holistic logic. Suppose that $\rightsquigarrow$ is classically equivalent to $\rightarrow$, i.e. material implication. Then we have the following*

- *(i) Every consistent $\mid\!\sim \in \mathcal{C}$ is non-monotonic.*
- *(ii) For any $\mid\!\sim \in \mathcal{C}$, Σ_σ is classically inconsistent.*
- *(iii) If $\mathcal{L}$ is finite dimensional, then Σ_g is classically inconsistent. In fact, it contains a classical contradiction.*

Proof. (i) and (ii) summarise results proved earlier.

As to (iii) let $(\mid\!\sim_i), i = 1, .., n$ a basis with pointers σ_i. We have by Theorem 24

$$\Sigma_g \vdash \neg\sigma_i,\ i = 1, ..., n.$$

Therefore

$$\Sigma_g \vdash \bigwedge \neg\sigma_i$$

For any $\mid\!\sim \neq 0$ we have by the definition of a basis

$$\not\mid\!\sim \bigwedge \neg\sigma_i$$

For otherwise $\mid\!\sim$ would be orthogonal to all elements of the basis contrary to the definition of a basis. It follows by Lemma ? that

$$\bigwedge \neg\sigma_i \rightsquigarrow \bot \in \Sigma_g$$

Thus

$$\Sigma_g \vdash \bigwedge \neg\sigma_i \rightsquigarrow \bot$$

and, since $\rightsquigarrow$ is classically equivalent to $\rightarrow$,

$$\Sigma_g \vdash \bigwedge \neg\sigma_i \to \bot$$

It follows that

$$\Sigma_g \vdash \bot$$

Thus Σ_g is classically inconsistent. Then there exists a finite set $\{\alpha_1, ..., \alpha_n\} \subset \Sigma_g$ which is classically inconsistent. Since Σ_g is closed under conjunctions we have

$$\bigwedge \alpha_i \in \Sigma_g$$

But this conjunction is a classical contradiction.

▮

We do not have the following yet in the book. It's in the section "Kochen-Specker-Schuette revisited" But the results are correct.

Corollary 9.1 *Let H be a finite dimensional orthomodular space and $dim H \geq 2$. Let $\mathcal{L}_{H,\psi}$ be a logic presented by H. Then Σ_g is classically inconsistent.*

Corollary 9.2 *Under the above hypotheses there exists a classical tautology ϕ such that for all $x \in H$ we have $\vdash_x \neg\phi$.*

We may strengthen the above resuls a bit. Namely note that in a Hilbert space H at least 2 we have the following. For any $x \in H$ there exists $y \in H$ such that neither $x \in \langle y \rangle$ nor $x \in \langle y \rangle^\perp$ Analysing the proof of the no windows theorem we then get

Corollary 9.3 *Let H be a Hilbert space logic of dimension at least 2. Let Σ_p be the set of formulas of Σ_g built up from pointers only. Then Σ_p is classically inconsistent. Thus there exists a classical tautology ϕ built up from pointers such that $\vdash_x \neg\phi$ for all $x \in H$.*

For the corollaries observe that logics presented by finite dimensional orthomodular spaces and thus by finite-dimensional Hilbert spaces are holistic and the internalising connective is the Sasaki hook, which is classically equivalent to material implication. Corollary 2 is the result by Kochen-Specker saying that there exists a classical contradiction which under a suitable interpretation of the variables as closed subspaces of a finite-dimensional Hilbert space represents the full space. It is even a generalisation, since the Kochen-Specker-Schuette tautology is in Hilbert space.

9.4 Limiting case theorem

The special theory of relativity has classical mechanics as a limiting case in the sense that classical mechanics holds for 'small velocities'. For 'small velocities' relativistic effects are negligible.

Similarly, quantum mechanics may be regarded as having classical mechanics as a limiting case. We investigate here the logical analogue of this fact within the framework of holistic logics.

9.4.1 Non-commuting operators in consequence revision sytems

Proposition 9.7 *Given a consequence revision system $\langle \mathcal{C}, F \rangle$. Suppose that all revision operators commute. Then every $\mid\!\sim$ is monotonic.*

Proof. Assume that all revision operators commute and let $\mid\!\sim \in \mathcal{C}$. Assume $\mid\!\sim \beta$. This means $\mid\!\sim_\beta = \mid\!\sim$. Now let α be any formula. Note that $\mid\!\sim_{\alpha,\beta} \beta$. The above notation means that $\mid\!\sim$ is first revised by α and then by β. Since the revision operators coresponding to α and β commute, we have $\mid\!\sim_{\beta,\alpha} = \mid\!\sim_{\alpha,\beta}$. But $\mid\!\sim_{\beta,\alpha} = \mid\!\sim_\alpha$. It follows that $\mid\!\sim_\alpha \beta$ but this says that $\alpha \mid\!\sim \beta$.Thus $\mid\!\sim$ is monotonic. ∎

Note that from the above proposition it follows that in a consequence revision system containing non-monotonic consequence relations we have non-commuting revision operators.

Let us now point out certain analogies between uncertainty relations and the projection postulate in quantum mechanics on the one hand and non commuting revision operators and non-monotonic consequence relations in consequence revision systems on the other.

Consider two observables A and B in quantum mechanics with non-commuting operators, say position and momentum. For such observables we have an uncertainty relation. Roughly, this means the following. Assume the quantum system is in a state x in which observable A is not sharp. In other words, if we measure A in state x, we get the values A which can assume only with a certain probabilities. Now, assume we perform a measurement of observable A and get as a result of measurement a certain value λ, i.e. the result of the measurement is a proposition of the form "$A = \lambda$". Now, according to quantum mechanics, the system has changed its state in such a way that in the new state y observable A is sharp and any subsequent measurement yields as a result the proposition "$A = \lambda$". This is the *projection postulate* of quantum mechanics. Note the analogy with revision! The state y, however, needn't be a state in which observable By is sharp. Assume we now perform a measurement of B. We then end up in a state z in which B is sharp with proposition "$B = \mu$". However, it may happen that in state z observable A is no longer sharp, i.e. the proposition "$A = \lambda$" no longer 'holds'. This phenomenon is due to the uncertainty relation holding between A and B. Note the analogy with non-monotonicity of consequence relations and revision!

We said that in quantum mechanics the passage to the limit, i.e. classical mechanics, is from uncertainty relations to the absence of uncertainty relations. What is, in view of the above intuitive consideration, the analogue of this process of passing to the limit at the level of logic? It should be the passage from non-monotonicity to monotonicity. Let's see what we get.

9.4.2 The Limiting Case Theorem

We call the following theorem the *Limiting Case Theorem* for holistic logics.

Theorem 9.10 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic such that every $\mid\!\sim \in \mathcal{C}$ is monotonic. Then $\mathcal{L}$ is totally degenerate, i.e. every $\mid\!\sim \in \mathcal{C}$ has the form $\vdash_\Sigma$ for some (consistent) complete classical theory Σ.*

Proof. Suppose that $\mid\!\sim \in \mathcal{C}$ is monotonic. Let α be such that $\not\mid\!\sim \alpha$. Then we have by 'provability of unprovability' that $\mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$ and by monotonicity $\alpha \mid\!\sim \neg(\sigma \rightsquigarrow \alpha)$. Since $[\neg(\sigma \rightsquigarrow \alpha)] = \{\mid\!\sim, 0\}$, we have $\mid\!\sim \neg\alpha$. Thus for any α we have either $\mid\!\sim \alpha$ or $\mid\!\sim \neg\alpha$. We say that $\mid\!\sim$ is complete as a consequence relation.

Put $\Sigma =: \{\alpha \mid\mid\!\sim \alpha\}$. We now observe that Σ has the following properties:

$$\text{(i) } \alpha \wedge \beta \in \Sigma \text{ iff } \alpha \in \Sigma \text{ and } \beta \in \Sigma$$

$$\text{(ii) } \alpha \vee \beta \in \Sigma \text{ iff } \alpha \in \Sigma \text{ or } \beta \in \Sigma$$

$$\text{(iii) } \neg\alpha \in \Sigma \text{ iff } \alpha\Sigma$$

It is then a fact of classical logic that Σ is classically consistent and complete.

(i) follows from the conditions we require a class of consequence relations to satisfy. We get (ii) as follows. $\mid\!\sim \alpha \vee \beta$ implies $\mid\!\sim \neg(\neg\alpha \wedge \neg\beta)$. This is one of the general conditions we impose on consequence relations in this book. By completeness of $\mid\!\sim$ we have that $\not\mid\!\sim (\neg\alpha \wedge \neg\beta)$. It follows that $\not\mid\!\sim \neg\alpha$ or $\not\mid\!\sim \neg\beta$ and thus again by completeness of $\mid\!\sim$, $\mid\!\sim \alpha$ or $\mid\!\sim \beta$. (iii) expresses completeness of $\mid\!\sim$

From the above it follows that $\alpha \mid\!\sim \beta$ iff $\mid\!\sim \neg\alpha$ or $\mid\!\sim \beta$. Namely, we have $\alpha \mid\!\sim \beta$ iff $\mid\!\sim_\alpha \beta$. Moreover, $\mid\!\sim_\alpha = \mid\!\sim$ iff $\alpha \in \Sigma$ iffr $\mid\!\sim_\alpha = 0$ iff $\neg\alpha \in \Sigma$. It follows that $\neg\alpha \in \Sigma$ or $\beta \in \Sigma$ or equivalently $(\alpha \to \beta) \in \Sigma$. But this means that $\to$ is internalising. We have proved that $\mid\!\sim = \vdash_\Sigma$ ∎

The following is another description of the 'limiting case'.

Theorem 9.11 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic. Then the following conditions are equivalent.*

- *Every $\mid\!\sim \in \mathcal{C}$ is monotonic.*
- *$\mathcal{L}$ is totally degenerate, i.e, every $\mid\!\sim \in \mathcal{C}$ is of the form $\mid\!\sim = \vdash_\Sigma$ for some complete classical theory Σ*
- *All revision operators commute*
- *$\mathcal{L}$ is one-dimensional.*

Another equivalent condition is that $\mathcal{L}$ is a one-dimensional Hilbert space logic. But at this stage we do not have this concept yet see Chapter 8).

Note that 'in the limit' we have monotonicity , classical consistency and thus *models*, a 'reality outside the logic'.

9.5 Reflecting on Self-Referential Completeness

Let us at this point reflect a bit on the salient features of holistic logics and their interplay. We are especially interested in the role of self-referential completeness.

What we, for the purposes of this and the next chapter of this book, called a logic is a triple of the form $\langle \mathcal{C}, F, \rightsquigarrow \rangle$. It has its motivation in a structure arising in a natural way in connection with classical logic (see section 9.1) The first ingredient of such a logic is a set $\mathcal{C}$ of logical entities called consequence relations. A consequence relation may, traditionally, be viewed as a logic in its own right. So, according to traditional terminology a logic in this sense is a set of logics. The intuition here is, however, that these various logics that constitute a holistic logic should not form too heterogeneous a collection. We may interpret the requirement that all these various consequence relations have a common internalising connective $\rightsquigarrow$ as expressing the intuition that $\mathcal{C}$ is not too heterogeneous. The third ingredient is that of an action F on $\mathcal{C}$. Formulas *act* on $\mathcal{C}$, and in general, this action is proper, i.e. given a formula α and some $\mid\!\sim \in \mathcal{C}$ we have in general that $\mid\!\sim_\alpha =: F(\alpha, \mid\!\sim) \neq \mid\!\sim$.

Now, what are the salient features of holistic logics?

We have seen that the consequence relations of a holistic logic are non-monotonic and self-referentially sound and and complete. Moreover, we have the No Windows Theorems for holistic logics. Intuitively, this says that in a sense, the consequence relations of a holistic logic are completely self-contained logical entities, logical monads so to speak. This is a remarkable phenomenon which - to the authors- is unfamiliar from other branches of logic. We, moreover, consider it remarkable that it is in connection with logical investigations into quantum mechanics that we hit upon this phenomenon.

Another salient feature of holistic logics is non-monotonicity: all their consequence relations are-in the non-denerate case- non-monotonic.

9.5.1 How an agent with full introspection can be consistent

Start here with remrks on Smullyan "Forever Undecided" In his two materpieces of logical writing **??** and **??** Smullyan introduces the concept of a self-referential system in connection with systems of modal logic. Given a modal system M with the modal operator $\Box$ and let $\vdash_M$ denote deducibility in the system M. Call a system M self-referentially sound if $\vdash_M \Box\alpha$ implies $\vdash_M \alpha$. Call it self$_referentiallycompleteif \vdash_M$ α implies $\vdash_M \Box\alpha$. Smullyan presents several systems of modal logic that are self-referentially sound, but he does not present a self-referentially complete modal system.

Make remarks here about the system G and the system D, which is in fact self-refrentially complete. But emphasise that Smullyan is dealing with the deictic parts of the system and the system D is the limiting classical case where the modal operator is trivial (invisible). We would like to make the point that it is non-monotonicity that plays a vital role in making (consistent) self-referentially complete systems possible.

The simple statement below -formulated as a theorem- expresses an elementary fact which, however, sheds light on the interplay between non-monotonicity, self-referential completeness and revision in holistic logics.

Technically, we use the term proof operator in the next theorem. For this recall that, in a holistic logic, we may view the meta language of a consequence relation as defined in [**?**] as a sublanguage of the object language. If we use $\Box\alpha$ as an abbreviation for $\sigma \rightsquigarrow \alpha$, then $\Box$ plays the role of a proof operator.

Definition 9.14 *Given a consequence relation $\mid\!\sim$. Suppose the language contains a connective (primitive or definable by other connectives) such that we have*

- $\mid\!\sim \alpha$ *iff* $\mid\!\sim \Box\alpha$
- $\not\mid\!\sim \alpha$ *iff* $\mid\!\sim \neg\Box\alpha$

Then we call $\Box$ a proof operator for $\mid\!\sim$.

The reader should note that in our previous investigations the existence of a proof operator was the source of self-referential completeness. Recall that given a holistic logic $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ and $\mid\!\sim \in \mathcal{C}$ with pointer σ. Then the proof operator $\Box$ is given by $\Box\alpha =: \sigma \rightsquigarrow \alpha$.

Theorem 9.12 *Given a consequence relation $\mid\!\sim$. Suppose $\mid\!\sim$ admits an internalising connective $\rightsquigarrow$ and a proof operator $\Box$. Then $\mid\!\sim$ has no proper consistent extension with $\rightsquigarrow$ as an internalising connective $\Box$ as a proof operator.*

Corollary 9.4 *Given a logic $\langle \mathcal{C}, F, \rightsquigarrow \rangle$, let $\mid\!\sim \in \mathcal{C}$. Suppose $\mid\!\sim$ admits a contingent proposition α, i.e. there exists a formula α such that $\not\mid\!\sim \alpha$ and $\not\mid\!\sim \neg\alpha$. Then $\mid\!\sim_\alpha$ admits a proof operator only if there exists a formula β such that $\mid\!\sim \beta$ and not $\mid\!\sim_\alpha \beta$.*

Note that the above theorem also holds in the limiting case of a consequence relation of the form $\vdash_\Sigma$ where Σ is a complete classical theory. The reason is that a complete classical theory is also maximal (see Chapter 2). Note that the corollary expresses non-monotonicity. The reader may view the argument in the proof ot theorem 9.12 given below as a formalisation of the argument we gave for the claim that an autoepistemic reasoner with positive and negative introspection must be non-monotonic.

Proof.

Let $\mid\!\sim$ be as in the theorem. Suppose there exists some $\mid\!\sim_0$ which is consistent and has $\rightsquigarrow$ as an internalising connective and $\Box$ as a proof operator and which properly extends $\mid\!\sim$, i.e. we have $\mid\!\sim \subset \mid\!\sim_0$ and this inclusion is proper. This means that there exist formulas α and β such that $\alpha \not\mid\!\sim \beta$ and $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim_0 \beta$. It follows that $\not\mid\!\sim \alpha \rightsquigarrow \beta$ and $\mid\!\sim_0 \alpha \rightsquigarrow \beta$. We then have that $\mid\!\sim \neg\Box(\alpha \rightsquigarrow \beta)$ and since $\mid\!\sim_0$ is an extension of $\mid\!\sim$ that $\mid\!\sim_0 \neg\Box(\alpha \rightsquigarrow \beta$. But on the other hand we have, since $\Box$ is assumed to be a proof operator for $\mid\!\sim_0$, that $\mid\!\sim \Box(\alpha \rightsquigarrow \beta)$, which contradicts the consistency of $\mid\!\sim_0$.

We get the corollary as follows. Since $\not\mid\!\sim \neg\alpha$ we have $\mid\!\sim_\alpha \neq 0$, i.e. $\mid\!\sim_\alpha$ is consistent. We have $\mid\!\sim_\alpha \alpha$. Thus $\mid\!\sim \neq \mid\!\sim_\alpha$. Assume $\mid\!\sim_\alpha$ admits a proof operator. Then, by the above theorem, $\mid\!\sim_\alpha$ is not an extension of $\mid\!\sim$. This means that there exists a formula β such that $\mid\!\sim \beta$ and not $\mid\!\sim_\alpha \beta$.

▮

Let us now think of the formal logial entity of a consequence relation as a sort of state (of mind) of some agent and view the proof operator as an epistemic operator. Think of $\Box\alpha$ as saying "I know α" and $\neg\Box\alpha$ as saying "I don't know α". Now consider a state of the agent in which he is undecided about some proposition α, i.e. he neither knows that α nor that $\neg\alpha$. In this case the reasoner can assign a 'truth value' neither to α nor to $\neg\alpha$, i.e his knowledge is incomplete. But let us assume that the agent has the possibility of learning that α, i.e. of 'completing' his knowledge. As a result of completing his knowledge by learning α he would, clearly, end up in a new state of mind, since he now knows α which in his previous ste he had not known. What the above considerations show is that this process of learning cannot be cumulative in the sense that he now knows more than he knew before. Rather the change of state must be such that he must have 'forgotten' something. There must be a proposition β he had known before but does not know in his new state of knowledge or of mind if you like. The transition of states in the process of learnin α must have been non-monotonic. It is the non-monotonic nature of the transition of states that saved his consistency.

9.5.2 The invisible proof operator in classical logic and classical mechanics

Let us now revisit the structures we considered in ?. Given a consistent set of formula Σ

Proposition 9.8 *Let Σ be consistent. Then $\vdash_\Sigma$ admits a (definable) proof operator iff Σ is complete (and thus maximal consistent). In this case the proof operator is trivial, namely $\Box\alpha = \top \to \alpha$.*

Proof. Suppose $\vdash_\Sigma$ admits a definable proof operator $\Box$. Then we have by the definition of a proof operator and by lassical logic that for any $\alpha \vdash_\Sigma \Box\alpha \leftrightarrow \alpha$ and thus $\vdash_\Sigma \neg\Box\alpha \to \neg\alpha$. Suppose that not $\Sigma \vdash \alpha$ which says that not $\vdash_\Sigma \alpha$. It follows that $\vdash_\Sigma \neg\Box\alpha$ and by the above remark $\vdash_\Sigma \neg\alpha$. This says that $\Sigma \vdash \neg\alpha$. Σ is complete and by lemma a maximal consistent set. It is obvious that $\top\alpha$ defines a (trivial) proof operator.

▮

Intuitively, we may view the simple facts stated above in the following light. As we saw several times earlier in the book, families of the form $\vdash_{\Sigma_i}{}_{i\in I}$, Σ_i is a complete classical theory arise as limiting cases of genuine holistic logics having a non-trivial proof operator. In the limiting case the proof operator applied to a formula α, i.e. the formula $\Box\alpha$ 'collaoses' to α. In the limiting case the statement

"α is provable$or, say,$"α is measurable$collapsessimplycollapsestotheassertionof\alpha$. Provability or measurability becomes just truth. The proof operator in the limiting case becomes invisible so to speak.

The gist of the above simple consideration is this. Suppose we want to the build a logic $\mathcal{C}, F, \rightsquigarrow$ such that all the elements of $\mathcal{C}$ are 'logical monads', i.e. logical entities which are self- referentially complete and display the 'no windows phenomenon'. Then the above says that, first, α must act properly, i.e. it must 'change' $\mid\!\sim$. Moreover, this change is non-monotonic in the sense that at least one formula will be 'lost' in that even if it was provable 'before the change' it is no longer provable 'after the change'. This is strikingly reminiscent of the uncertainty relations in Quantum Mechanics where things are as follows. Given a state of a physical system in which the physical quantity A is not sharp. Then measure A. As a result we end up in a state in which A is sharp, say $A = \mu$. But this has a price. There must be some physical quantity B which was sharp before measurement and which is no longer sharp after measurement. Note that, logically in view of the above consideration the source of this phenomenon is the existence of a proof operator, which essentially amounts to self-referential completeness.

REFLECT HERE ON THE NOTION OF STATE.

9.5.3 Feynman on the uncertainty principle: the logical tightrope

The Feynman lectures are unique among the textbooks on physics in many respects. Let us see what Feynman says about Heisenberg's uncertainty principle. Here are some quotations. "Let us show for one particular case that the kind of relation given by Heisenberg must be true in order to keep from getting into trouble." What does he mean by 'getting into troble? To get a clearer picture, here is another quotation: "The uncertainty principle protects quantum mechanics." What does the uncertainty principle 'protect' quantum mechanics from? Again a quotation: $...ifawaytobeattheuncertaintyrelationwereeverdiscovered, wouldgiveinconsistentresu$
$inconsistency!Theuncertaintyprincipleprotectsquantummechanicsfrominconsistency.Inconsister$
$"Thisisthelogicaltightropeonwhichwemustwalkifwewishtodescribenaturesuccessfully".Quantum$
$accordingtoFeynman - guaranteetheconsistencyofthesystem.$

Chapter 10

Towards Hilbert Space

Let us at this point come back to the general idea underlying the enterprise of this book. In Chapter 7 we asked ourselves the question what logic could do about Quantum Mechanics. We arrived at the conclusion that it would not be our aim to find a new deductive system especially suited for reasoning about Quantum Mechanics. Rather we said that we should look for logical structures implicit in the formalism of Quantum Mechanics which could prove useful in the task of trying to understand this very formalism. We isolated two types of (related) structures: M-algebras and holistic logics. Both are abstractions from structures we find in Hilbert space. In this we go further. We ask the question "Can we characterise the core concept of the formalism, namely that of a Hilbert space in terms of these structures?" This is part of what in the literture on the foundations of Quntum Mechanics is sometimes called the representation enterprise, and theorems of this sort are regarded as 'derivations' of the formalism of Quantum Mechanics from first principles. What we offer in this Chapter is such derivation. Its peculiarity is that the first principles from which we start are purely logical in nature.

10.1 Presenting Holistic Logics

10.1.1 Orthomodular Holistic logics

Let H be an orthomodular space. Recall that $Sub(H)$ denote the the set of closed subspaces of H. We know known that $\langle Sub(H), \subset, ^{\perp} \rangle$ is an orthomodular lattice. Recall that that $^{\perp}$ means orthogonal complement formation. We shall use capital letters $A, B, ...$ for subspaces and, if there is no danger of confusion, for the corresponding projectors. Moreover, we use the symbols for Boolean connectives in connection with closed subspaces, i.e we write $A \wedge B$ for $A \cap B$ and we denote the smallest closed subspace containing the closed subspaces A and B by $A \vee B$.
Let Fml be a propositional anguage, i.e. closed under $\neg$, $\wedge$ and containing $\top$ and $\perp$ and let $\Psi : Fml \to Sub(H)$ be a surjective function such that $\Psi(\alpha \wedge \beta)) =$

$\Psi(\alpha) \wedge \Psi(\beta)$ and $\Psi(\neg\alpha) = \Psi(\alpha)^{\perp}$. Denote the projectiom corresponding to $\Psi(\alpha)$ by A. Let $x \in H$. Then we define the consequence relation $\vdash_x$ by

$$\alpha \vdash_{x,\Psi} \beta \text{ iff } Ax \in \Psi(\beta).$$

We shall simply write $\vdash_x$ if Ψ is clear from the context. Note that $\vdash_x$ depends only on the ray of x, i.e. $\vdash_{x_1}=\vdash_{x_2}$ iff the one dimensional subspace $\langle x_1 \rangle$ generated by $x_1$is equal to the one dimensional subspace $\langle x_2 \rangle$ generated by x_2.
Given an orthomodular space H and a function Ψ as described above, we define

$$\mathcal{C}_{H,\Psi} =: \{\vdash_x |\ x \in H\}.$$

Let us now define a function that will turn out to be an action on $\mathcal{C}_{H,\Psi}$. Define $\mathcal{F}_{H,\Psi} : Fml \times \mathcal{C}_{H,\Psi} \rightarrow \mathcal{C}_{H,\Psi}$ by

$$\mathcal{F}_{H,\Psi}(\alpha, \vdash_x) =: \vdash_{Ax}.$$

Note that $\mathcal{F}_{H,\Psi}$ is well defined, since $\langle x_1 \rangle = \langle x_2 \rangle$ implies $\langle Ax_1 \rangle = \langle Ax_2 \rangle$.
Recall that the Sasaki hook $\rightsquigarrow_s$ is the connective defined as follows: $\alpha \rightsquigarrow \beta = \neg\alpha \vee (\alpha \wedge \beta$.

Theorem 10.1 *Let H be an orthomodular space and Ψ a surjective function as described above. Then $\mathcal{L}_{H,\Psi} =: \langle \mathcal{C}_{H,\Psi}, \mathcal{F}_{H,\Psi}, \rightsquigarrow_s \rangle$ is a holistic logic. All consequence relations satisfy conditions 1 and 2 in chapter with the possible exception of Cut and Cautious Monotonicity. In case H is a Hilbert space all conditions are satisfied.*

The following proof is in case that H is a Hilbert space. Cut and Cautious Monotonicity work in the Hilbert space case only

Proof. We first need to verify the conditions imposed on the elements of $\mathcal{C}$. This is routine for the most part.
Reflexivity is a consequence of the fact that for $x \in \Psi(\alpha)$ we have $Ax = x$.
Let us first verify *Cut*. So let $x \in H$ and assume $\alpha \wedge \beta \vdash_x \gamma$ and $\alpha \vdash_x \beta$. $\alpha \vdash_x \beta$ says that $\Psi(\alpha)_x \in \Psi(\beta)$. Moreover, from the above assumptions it follows that $\Psi(\alpha \wedge \beta)_x = \Psi(\alpha)_x$. By the hypothesis we have $\Psi(\alpha \wedge \beta)_x \in \Psi(\gamma)$ and thus $\Psi(\alpha)_x \in \Psi(\gamma)$. But this means that $\alpha \vdash_x \gamma$. Thus, *Cut* is verified.
We now verify *Restricted Monotonicity.* Assume $\alpha \vdash_x \beta$ and $\alpha \vdash_x \gamma$. It follows that $\Psi(\alpha)_x = \Psi(\alpha \wedge \beta)_x$ and, since by the hypothesis we have $\Psi(\alpha)_x \in \Psi(\gamma)$, we see that $\Psi(\alpha \wedge \beta)_x \in \Psi(\gamma)$, which says that $\alpha \vdash_x \gamma$. Thus *Restricted Monotonicity* is verified.
In order to verify the other conditions use that by definition we have $\Psi(\alpha \wedge \beta) = \Psi(\alpha) \wedge \Psi(\beta)$ and $\Psi(\neg\alpha) = \Psi(\alpha)^{\perp}$ and elementary Hilbert space theory.
Verify here the 'global' conditions for $\mathcal{C}$ and make the properties of orthomodular spaces used explicit, pwrkaps in a previous lemma

For the first global condition for instance suppose $\alpha \mid\!\sim_{\mathcal{C}_{H,\Psi}} \gamma$ and $\beta \mid\!\sim_{\mathcal{C}_{H,\Psi}} \gamma$. This means $\Psi(\alpha) \subset \Psi(\gamma)$ and $\Psi(\beta) \subset \Psi(\gamma)$. It is then elementary Hilbert space theory that $\Psi(\alpha \vee \beta) \subset \Psi(\gamma)$. But this says that $\alpha \vee \beta \mid\!\sim_{\mathcal{C}_{H,\Psi}} \gamma$.

We now prove that $\mathcal{F}_{H,\Psi}$ is an action on $\mathcal{C}$. Condition (i) in the definition of an action is obvious. Consider condition *(ii)* in Definition ?. Suppose $\vdash_x \neg\alpha$. This is equivalent to $x \in \Psi(\alpha)^\perp$, which is the case iff $Ax = 0$. But this means $\vdash_{Ax} = \mathcal{F}_{H,\Psi}(\alpha, \vdash_x) = 0$.
Consider condition *(iii)* in the definition of an action. Let $\mathcal{F}_{H,\Psi}(\beta, (\mathcal{F}_{H,\Psi}(\alpha, \vdash_x)) = \mathcal{F}_{H,\Psi}(\alpha), \vdash_x)$. This is the case iff $BAx = Ax$, which is equivalent to $Ax \in \Psi(\beta)$. But this says that $\alpha \vdash_x \beta$.

We still need to prove that $\rightsquigarrow_s$ is internalising for $\mathcal{C}$. Suppose $\alpha \vdash_x \beta$. By definition this means that $Ax \in \Psi(\beta)$. By Lemma ? this is the case iff $x \in \neg A \vee (A \wedge B)$. But this says $\vdash_x \alpha \rightsquigarrow_s \beta$. ∎

We call a logic of the above form an *orthomodular logic*. In case case H is a classical Hilbert space (see Chapter 3) we call $\mathcal{L}_{H,\Psi}$ a *Hilbert space logic.*

Note that in the context of an orthomodular logic the rays of the underlying orthomodular space H have a precise logical meaning, namely as representing consequence relations, which, metaphorically speaking, can be viewed as 'states of provability'.

10.1.2 The Canonical $\mathcal{H}$-Model for a Hilbert Space Logic

Definition 10.1 *Let H be a Hilbert space, Ψ a function as described in the last section and $x \in H$. Define the binary relation $\leq_x$ on H as follows*

$$x_1 \leq_x x_2 \text{ iff: } d(x, x_1) \leq d(x, x_2)$$

Moreover, define the structure

$$\mathcal{M}_{x,\Psi} = \langle H, \leq_x, l_\Psi \rangle \ ,$$

as follows. Let $x \in H$, then $l_\Psi(x) = \{s_x\}$ is the singleton consisting of the following Scott-model s_x: For $\alpha \in Fml$ put $s_x(\alpha) = 1$, if $x \in \Psi(\alpha)$, else $s_x(\alpha) = 0$.

Lemma 10.1 *Let $\mathcal{L}_{H,\Psi}$ be a Hilbert space logic. Then for every $x \in H$, $\mathcal{M}_{x,\Psi}$ is a $GKLM$ model for $\vdash_{x,\Psi}$.*

Proof. We first have to verify the smoothness condition. For this observe that for any α we have $[\alpha] = \Psi(\alpha)$. Note that the notation $[\alpha]$ is in the sense of definition of a $(GKLM)$. It suffices to show that every $[\alpha]$ has a unique $\leq_x$-minimal element. But this is what Proposition 10 says, namely Ax is that unique minimal element.
It remains to be shown that $\vdash_{x,\Psi} = \vdash_{\mathcal{M}_{x,\Psi}}$. So let $\alpha \vdash_{x,\Psi} \beta$. By definition this means $Ax \in [\beta]$. But this is equivalent to $\alpha \vdash_{x,\Psi} \beta$, since Ax is the minimal element of $[\beta]$. ∎

We now define an $\mathcal{H}$-structure for a given Hilbert space logic $\mathcal{L}_{H,\Psi}$.

Definition 10.2 *Given the Hilbert space logic $\mathcal{L}_{H,\Psi} = \langle \mathcal{C}_{H,\Psi}, \mathcal{F}_{H,\Psi}, \rightsquigarrow_s \rangle$. Consider the structure $\mathcal{H}_{H,\Psi} = \langle H, h, \mathcal{F}, l_\Psi, g \rangle$ such that*

- $h(x) = \vdash_x$
- $\mathcal{F}(\alpha, x) = Ax$
- *The function* l_Ψ *is defined as follows:* $l_\Psi(x) = \{s_x\}$, *where* $s_x(\alpha) = 1$ *if* $x \in \Psi(\alpha)$, 0 *else.*
- $g(x) = \leq_x$ *as defined in Definition* 9.

Theorem 10.2 *Given a Hilbert space logic* $\mathcal{L}_{H,\Psi} = \langle \mathcal{C}_{H,\Psi}, \mathcal{F}_{H,\Psi}, \rightsquigarrow_s \rangle$. *Then* $\mathcal{H}_{H,\Psi}$ *as defined above is an* $\mathcal{H}$*-model for* $\mathcal{L}_{H,\Psi}$.

10.1.3 Hilbert space logics as holistic logics: some properties

Lemma 10.2 *Let* $\mathcal{L}_{H,\Psi}$ *be a Hilbert space logic and* $x \in H$ *non zero. Then*

- *(i) For every* x' *not orthogonal to* x *we have* $\sigma_x \vdash_{x'} \alpha \rightsquigarrow_s \beta$ *iff* $\alpha \vdash_x \beta$
- *(ii)* $\vdash_x \alpha$ *iff* $\Psi(\sigma_x \rightsquigarrow_s \alpha) = H$ *and thus* $\Psi(\neg(\sigma_x \rightsquigarrow_s \alpha)) = \{0\}$
- *(iii) not* $\vdash_x \alpha$ *iff* $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp$ *and thus* $\Psi(\neg(\sigma_x \rightsquigarrow_s \alpha)) = \langle x \rangle$

Proof. (i) We have by elementary Hilbert space theory that $\overline{\sigma}_x \vdash_{x'} = \vdash_x$ if x' is not orthogonal to x, else $\overline{\sigma}_x \vdash_{x'} = 0$. Suppose that x' is not orthogonal to x and $\sigma_x \vdash_{x'} \alpha \rightsquigarrow_s \beta$. By the above remark this is equivalent to $\vdash_x \alpha \rightsquigarrow_s \beta$ and, since $\rightsquigarrow_s$ is internalising, this is the case iff $\alpha \vdash_x \beta$. This proves (i).
(ii) Recall that $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp \vee (\langle x \rangle \wedge \Psi(\alpha))$. $\vdash_x \alpha$ means that $\langle x \rangle \wedge \Psi(\alpha) = \langle x \rangle$. We thus have $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp \vee \langle x \rangle = H$.
(iii) not $\vdash_x \alpha$ means that $\langle x \rangle \wedge \Psi(\alpha) = \{0\}$ and thus $\Psi(\sigma_x \rightsquigarrow_s \alpha) = \langle x \rangle^\perp$. ∎

We call α x-consistent iff not $\vdash_x \neg\alpha$.

Theorem 10.3 *Let* $\mathcal{L}_{H,\Psi}$ *be a Hilbert space logic and* $x \in H$ *non zero. Then we have*

$$\text{(i) } \vdash_x \alpha \text{ iff } \vdash_x \sigma_x \rightsquigarrow_s \alpha$$

$$\text{(ii) not } \vdash_x \alpha \text{ iff } \vdash_x \neg(\sigma_x \rightsquigarrow_s \alpha)$$

$$\text{(iii) } \alpha \text{ is } x\text{-consistent iff } \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$$

Let α *be* x*-consistent.Then we have*

$$\text{(iv) } \vdash_x \alpha \text{ iff } \alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$$

Proof. (i) and (ii) follow immediately from Lemma 10. As to (iii) note that the x-consistency of α means that not $\vdash_x \neg\alpha$ and thus (ii) implies (iii).
For (iv) suppose $\vdash_x \alpha$. This means that $Ax = x$. Since α is x-consistent, we have $\Psi(\neg(\sigma_x \rightsquigarrow_s \neg\alpha)) = \langle x \rangle$ and thus $Ax \in \Psi\neg(\sigma_x \rightsquigarrow_s \neg\alpha))$, which by definition means $\alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$.

For the other direction assume $\alpha \vdash_x \neg(\sigma_x \rightsquigarrow_s \neg\alpha)$. Since α is x-consistent, We have $Ax \neq 0$. Moreover, since $\neg(\sigma_x \rightsquigarrow_s \neg\alpha) = \langle x \rangle$, x is an eigenvalue of A. Using the fact that the only eigenvalues of the projector A are 1 and 0, we get that $Ax = x$ and thus $\vdash_x \alpha$. ∎

10.2 Kochen-Specker-Schütte revisited

10.2.1 Classical inconsistency in Hilbert space logics

In this section wecome back to the phenomenon first observed by Kochen, Schütte and Specker, namely that Birkoff-von Neumann quantum logic is 'classically inconsistent'.

We start with the following observation. We denote by H_n n-dimensional Hilbert space. Let x_1, x_2 be non-orthogonal and non-collinear vectors of H_2.. Let Fml be the language of propositional logic and consider a Hilbert space logic $\mathcal{L}_{H_2,\Psi_0}$ such that for the propositional variables p_1, p_2 we have $\Psi_0(p_i) = \langle x_i \rangle$, $i = 1, 2$. Consider the formula $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ such that

$$\phi_1 = p_1 \vee p_2$$

$$\phi_2 = \neg p_1 \vee p_2$$

$$\phi_3 = p_1 \vee \neg p_2$$

$$\phi_4 = \neg p_1 \vee \neg p_2$$

It is easily seen that ϕ is a classical contradiction which is provable in all consequence relations of $\mathcal{L}_{H,\Psi_0}$.

Proposition 10.1 *ϕ is a classical contradiction and for all consequence relations $\mid\!\sim$ of $\mathcal{L}_{H_2,\Psi_0}$ we have $\mid\!\sim \phi$*

In the case of three dimensional Hilbert space H the above result is much more difficult to establish. In [3] Kochen and Specker gave a classical tautology the negation of which is provable in all consequence relations of a Hilbert space logic presented by three dimensional Hilbert space. It has 117 propositional variables.

Proposition 10.2 (Kochen-Specker) *There exists a classical contradiction α and a Hilbert space logic $\mathcal{L}_{H_3,\Psi}$ such that $\mid\!\sim \alpha$ for all $\mid\!\sim$ of $\mathcal{L}$.*

Here apply the sharpened no windows theorem and derive the above as a corollary. Comment on this!

This is no accident. It is a spcial case of the following thorem which in turn is an immediate consequence of the No Windows Theorem.

Theorem 10.4 *Let H be a finite-dimensional orthomudalar space of dimension at least 2. Then there exists a classical tautology which under a certain valuation of its variables as closed subspaces of H represents the zero space.*

Proof. USE THE GLOBAL NO WINDOWS THEOREM
∎

10.2.2 Birkhoff-von Neumann revisited

von Neumann's letter to Birkhoff

The following has already been said.

"Your general remarks I think are very true: I, too, think that our paper will not be very exhaustive or conclusive, but that we should not attempt to make it such: The subject is obviously only at the beginning of a development, and we want to suggest the direction of this development much more, than to reach final results. I, for one, do not even believe that the right formal frame for quantum mechanics is already found."

Birkhoff' s conditional

The following should probably be said in connection with Implication M-algebras.

von Neumann to Birkhoff: "Last spring you observed: Why not introduce a logical operation ab for any two (not necessarily simultaneously decidable) properties a and b, meaning this: If you first measure a you find that it is present, if you next measure b, you find that it is present too.

This ab cannot be described by any operator, and in particular not by a projection (= linear subspace). The only answer I could then find was this: There is no state in which the property ab is certainly present, nor is any in which it is certainly absent (assuming that a,b are sufficiently non-simultaneously definable = that their projection operators E,F have no common proper-functions not = 0 at all).

Of course, for this reason ab is no physical quantity relatively to the the machinery of quantum -mechanics. But how can one motivate this, how can one find a criterion of what is a physical quality and what not, if not by the 'causality' criterion: A statement describes if and only if the states in which it can be decided with certainty form a complete set.

I wanted to avoid this rather touchy and complicated question, and withdraw to the safe - although perhaps narrow- position of dealing with 'causal' statements' only. Do you propose to discuss the question fully? It might become too philosophical, but I would not say that I object absolutely to it. But it is dangerous ground-except you have a new idea , which settles the question more satisfactorily."

Let us take a closer look at this letter. In this letters JvN refers to a 'logical operation' proposed by Birkhoff in an earlier letter (spring 1935?). What does Birkhoff mean by 'logical operation'? A oonnective probably, so "ab" is a conditional in our sense. The conditional proposed by Birkhoff is similar to but different from ours. Our conditional says: "If you measure A, you end up in a state in which B is sharp." Think of "sharp" as "provable". So given a state x. Then after masuring A we are in state Ax. The requirement that in this state B

is sharp says that $BAx = Ax$. In this terminology, Birkhoff's conditional says: "If you first measure A and then measure B, you end up in a state in which A is still sharp.". Mathematically this means: $ABAx = BAx$.

Now JvN is not satisfied with this arguing that this 'operation' is not representable by a projection (subspace). The following is a reconstruction of his argument. Let A, B such that $A \cup B = \{0\}$, not $B \subset A^{\perp}$ and not $B \subset A$Such a constellation exists in every Hilbert space of dimension at least 3.

His first observation is : "There is no state in which the property ab is certainly present" What does this mean? For A, B as above, $ABAx = BAx$ implies $BAx = 0$ or equivalently $Ax \in B^{\perp}$

In our terminology, the consequence relation has no internalising connective definable by

Think about this.

We will come back to this Chapter later in "Birkhoff-von Neumann Revisited", where we we will establish the precise connection betweeen Birkhoff-von Neumann and the approach put forward in this book

10.3 Symmetry and Hilbert Space Presentability: The Representation Theorem

Note that the proof of the Representation Theorem given below can be simplified using our theorem characterising classical Hilbert lattices

In the previous sections we investigated some of the properties displayed by Hilbert space logics. In this section we are looking for properties characterising Hilbert space logics. To pose the problem more precisely, let us introduce the following terminology. Given a logic $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$, a Hilbert space H and a function $\Psi \rightarrow Sub(H)$ such that $\mathcal{L} = \mathcal{L}_{H,\Psi}$. Then we say that $\mathcal{L}$ is *presented* by H via Ψ. We say that $\mathcal{L}$ is presentable by H if there exists a function Ψ such that $\mathcal{L}$ is presented by H via Ψ. It is our aim to characterise the logics presentable by some Hilbert space H. In other words, we are looking for necessary and sufficient conditions for a logic $\mathcal{L}$ to be presentable by some Hilbert space H. We shall see that, besides some natural logical conditions, there are two properties essential for the characterisation we have in mind. The first property is what in section ? we called *holicity*. The second essential property, which we haven't come across yet, is a symmetry property. We shall call it the *symmetry property*. We shall see that, essentially, these two properties, namely *holicity* and *symmetry*, characterise Hilbert space logics. To be more precise, they characterise those logics which are presentable by infinite-dimensional Hilbert spaces, i.e. those structures playing a dominant role in quantum mechanics. Mathematically, the main pillar of our reasoning is Solèr's celebrated result on the characterisation of (infinite-dimensional) Hilbert spaces.

10.3.1 More about Holistic Logics

Several of the following things are treated earlier in the book or need to be treated

earlier.

Lemma 10.3 *Let $\mathcal{L}$ be a logic. Then any two pointers are $\mathcal{C}$-equivalent For a pointer σ to $\mid\!\sim_0$ we have for any $\mid\!\sim \in \mathcal{C}$ that $\mid\!\sim_\sigma = \mid\!\sim_0$ if $\not\mid\!\sim \neg\sigma$, otherwise $\mid\!\sim_\sigma = 0$.*

Given two consequence relations $\mid\!\sim_1$ and $\mid\!\sim_2$ with pointers σ_1 and σ_2 respectively. We say that $\mid\!\sim_1$ is *orthogonal* to $\mid\!\sim_2$ iff $\mid\!\sim_1 \neg\sigma_2$. This relation is symmetric and we say that the two consequence relations are orthogonal.

The above lemma belongs to section?

Lemma 10.4 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic, $\mid\!\sim_0 \in \mathcal{C}$ having pointer σ to itself. Then*

- *(i) For any $\mid\!\sim \in \mathcal{C}$ not orthogonal to $\mid\!\sim_0$ we have $\sigma \mid\!\sim (\alpha \rightsquigarrow \beta)$ iff $\alpha \mid\!\sim_0 \beta$*
- *(ii) $[\sigma] = \{\mid\!\sim_0, 0\}$*
- *(iii) $\sigma \mid\!\sim_{\mathcal{C}} (\alpha \rightsquigarrow \beta)$ iff $\alpha \mid\!\sim_0 \beta$ for any $\mid\!\sim$ not orthogonal to $\mid\!\sim_0$.*
- *(iv) For any non zero $\mid\!\sim_1, \mid\!\sim_2 \in \mathcal{C}$, $\mid\!\sim_1 \subset \mid\!\sim_2$ implies $\mid\!\sim_1 = \mid\!\sim_2$*

Proposition 10.3 *Hilbert space logics are holistic logics.*

This follows from the fact that Hilbert spaces are orthomodular spaces and the logics presented by orthomodular spaces are are holistic. This must in the final version be section "Presenting holistic logics"

The following proof is for Hilbert space logics.

Proof. Recall that, given a Hilbert Space H and $A \neq H, \{0\}$ a closed subspace, then there exists an $x \in H$ such that xA and not $xA^\perp$. From this it follows that Hilbert space logics are non trivial.

Let $\mathcal{L}_{H,\Psi}$ be any Hilbert space logic. For any non zero elements x, x' of H we need to show that $\overline{\sigma}_x \vdash_{x'} = \vdash_x$ or $\overline{\sigma}_x \vdash_{x'} = 0$. But this is equivalent to the following fact of elementary Hilbert space theory. Denote by I_x the projection operator corresponding to the ray $\langle x \rangle$. Then $I_x(\langle x' \rangle) = \langle x \rangle$ if x and x' are not orthogonal and $I_x(\langle x' \rangle) = \{0\}$ otherwise. ∎

Lemma 10.5 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow_s \rangle$ be a holistic logic. Then $\langle Fml, \leq, ^* \rangle$ and thus $\langle Prop, \subset, ^* \rangle$ are orthomodular, atomic and irreducible lattices.*

Proof. We have orthomodularity by the fact that $\mathcal{L}$ is a logic with the Sasaki hook as its internalising connective and Theorem 18. As to atomicity observe that the atoms of $\langle Prop, \subset, ^* \rangle$ are of the form $[\sigma_{\mid\sim}]$.

For irreducibility we need to prove that the center of that lattice consists of truth and falsity only. For this it suffices to prove that for every proposition $[\alpha]$ not representing truth or falsity there exists an atom $[\sigma_{\mid\sim}]$ such that $[\alpha]$ and $[\sigma_{\mid\sim}]$ are not compatible. In the special case of a pointer $\sigma_{\mid\sim}$ and a formula α compatibility says that $[\sigma_{\mid\sim}] \subset [\alpha]$ or $[\sigma_{\mid\sim}] \subset [\neg\alpha]$. Since $\mathcal{L}$ is non trivial,

for a given formula α there exists a $\mid\!\sim_o \in \mathcal{C}$ such that neither $[\sigma_{\mid\sim_0}] \subset [\alpha]$ nor $[\sigma_{\mid\sim_0}] \subset [\neg\alpha]$ and thus $[\alpha]$ and $[\sigma_{\mid\sim_0}]$ are not compatible. ▮

The following theorem gives a connection between local and global consequence in holistic logics. Its proof is routine.

Theorem 10.5 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a holistic logic. Let $\mid\!\sim \in \mathcal{C}$. Then the following statements are equivalent.*

- *(i)* $\mid\!\sim \alpha$
- *(ii)* $\sigma_{\mid\sim} \mid\!\sim_{\mathcal{C}} \alpha$
- *(iii)* $\mid\!\sim_{\mathcal{C}} (\sigma_{\mid\sim} \rightsquigarrow \alpha)$

Definition 10.3 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow \rangle$ be a logic.*

- *We say that $\mathcal{L}$ has the upward finiteness property, in brief the uf-property, iff the following holds: Given a set Σ of formulas. Then there exists a formula ψ such that $\sigma \mid\!\sim_{\mathcal{C}} \psi$ for every $\sigma \in \Sigma$ and the following condition is satisfied. For any formula ρ such that $\sigma \mid\!\sim_{\mathcal{C}} \rho$ for every $\sigma \in \Sigma$, we have $\psi \mid\!\sim_{\mathcal{C}} \rho$.*
- *We say that $\mathcal{L}$ has the downward finiteness property, in brief the df-property iff the following holds:*

  Given a set Σ of formulas. Then there exists a formula χ such that $\chi \mid\!\sim_{\mathcal{C}} \sigma$ for every $\sigma \in \Sigma$ and the following condition is satisfied. For any formula ρ such that $\rho \mid\!\sim_{\mathcal{C}} \sigma$ for every $\sigma \in \Sigma$, we have $\rho \mid\!\sim_{\mathcal{C}} \chi$.
- *In the case that $\mathcal{L}$ is holistic we say that $\mathcal{L}$ has the covering property iff the following condition is satisfied. Given a formula α and $\mid\!\sim \in \mathcal{C}$ such that $\not\mid\!\sim \alpha$. Then for any formula ρ such that $\alpha \mid\!\sim_{\mathcal{C}} \rho$ and $\rho \mid\!\sim_{\mathcal{C}} \alpha \vee \sigma_{\mid\sim}$ we have $\rho \equiv_{\mathcal{C}} \alpha \vee \sigma_{\mid\sim}$ or $\rho \equiv_{\mathcal{C}} \alpha$*

Intuitively we may think of the formulas ψ and χ in the above definition of playing the role of 'infinite disjunction' and 'infinite conjunction' of the formulas of Σ. The properties defined above are such that the following lemma holds.

Lemma 10.6 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow_s \rangle$ be a holistic logic having the df, uf and the covering properties. Then the lattices $\langle \overline{F}ml, \leq, ^* \rangle$ and thus $\langle Prop, \subset, ^* \rangle$ are orthomodular, atomic, irreducible, complete lattices having the covering property.*

10.3.2 Symmetry and Hilbert Space Logics

Let us start with the following observation. Let H be a Hilbert space and let $(x_i)_{i\in I}$ be a complete orthonormal system of H. Then any permutation of the system $(x_i)_{i\in I}$, more precisely any permutation of the index set I, induces a unique unitary transformation on H and thus an automorphism of the lattice $Sub(H)$. This fact reflects a symmetry property of Hilbert spaces and in view

of Solèr's theorem seems to be at the heart of the concept of a Hilbert space. It is the above fact that serves us as a motivation for the concept of a symmetric logic which we will study in the sequel.

Definition 10.4 *Let $\mathcal{L}$ be a holistic logic having the properties in the last lemma. Let $\Delta = (\mid\!\sim_i)_{i\in I}$ be an infinite family of consequence relations of $\mathcal{L}$ with the following properties*

- *(i) For $i \neq j$, $\mid\!\sim_i$ and $\mid\!\sim_j$ are orthogonal.*
- *(ii) For any consequence relation $\mid\!\sim$ of $\mathcal{L}$ there exists an $i_0 \in I$ such that $\mid\!\sim$ and $\mid\!\sim_{i_0}$ are not orthogonal.*

Then we call Δ a basis for $\mathcal{L}$.

Remark: Intuitively, we may think of a basis Δ of a holistic logic $\mathcal{L}$ as follows. Given any consequence relation $\mid\!\sim$ of $\mathcal{L}$. Then there exists a member of Δ in which $\mid\!\sim$ is encoded via the internalising connective. The system Δ may thus be viewed as containing the whole information of $\mathcal{L}$.

Definition 10.5 *Let $\mathcal{L}$ be a logic as in the last definition and let $\Delta = (\mid\!\sim_i)_{i\in I}$ be a basis for $\mathcal{L}$. We say that $\mathcal{L}$ satisfies the symmetry condition with respect to Δ iff the following holds. Let $f : I \to I$ be any permutation of the index set I. Then there exists an automorphism φ_f of the algebra of propositions of $\mathcal{L}$ (and thus of the algebra of operators) such that*

- *$\varphi_f([\sigma_i]) = [\sigma_{f(i)}]$, where $(\sigma_i)_{i\in I}$ ia any family such that σ_i is a pointer to $\mid\!\sim_i$.*
- *If the subset $J \subset I$ of those elements of I that are left fixed by f is non empty, then φ_f induces the identity on $[0, A]$, where A is the smallest proposition containing $[\sigma_j]$ for all $j \in J$.*

We say that $\mathcal{L}$ satisfies the symmetry condition (synonymously: is symmetric) iff there exists a basis Δ for $\mathcal{L}$ such that $\mathcal{L}$ is symmetric with respect to Δ.

Recall the notation $[0, A]$. It is the set of all propositions smaller than or equal to A eqiuipped with a lattice structure in a natural way. In the following theorem we assume the 'presenting' function Ψ to be surjective.

Theorem 10.6 *Let $\mathcal{L} = \langle \mathcal{C}, F, \rightsquigarrow_s \rangle$ be a logic. Then the following conditions are equivalent.*

- *(i) $\mathcal{L}$ is symmetric.*
- *(ii) There exists an infinite-dimensional classical Hilbert Space H presenting $\mathcal{L}$.*

Proof. For the direction from (ii) to (i) assume that there exists an infinite-dimensional classical Hilbert space H and a (surjective) function Ψ such that $\mathcal{L} = \mathcal{L}_{H,\Psi}$. We need to verify the symmetry property. Let $(x_i)_{i\in I}$ be a complete orthonormal system of H. Then $\Delta = (\vdash_{x_i})_{i\in I}$ is a basis for $\mathcal{L}$. Now observe that the lattice of propositions of $\mathcal{L}$ and $Sub(H)$ are isomorphic in a canonical way, namely via $[\alpha] \mapsto \Psi(\alpha)$. Thus, for the proof of symmetry it suffices to establish the following. For any permutation $f : I \to I$ there exists an automorphism ρ_f of $Sub(H)$ with the following properties:

- $\rho(\langle x_i\rangle) = \langle x_{f(i)}\rangle$
- If the set $J = \{i \mid f(i) = i\}$ is non-empty, then ρ_f induces the identical map on $[0, X]$, where X denotes the smallest closed subspace of H containing $\langle x_j\rangle$ for all $j \in J$.

To verify the above, recall that any for any $x \in H$ we have $x = \sum_{i\in I}\langle x, x_i\rangle x_i$. Define the map φ_f as follows. For $x = \sum_{i\in I}\langle x, x_i\rangle x_i$ put $\varphi_f(x) = \sum_{i\in I}\langle x, x_{f^{-1}(i)}\rangle x_i$. φ_f is well defined. We have for any $i \in I$ that $\varphi_f(x_i) = x_{f(i)}$. Moreover, φ_f is unitary, since for any $x, y \in H$ we have $\langle\varphi_f(x), \varphi_f(y)\rangle = \sum_{i\in I}\langle x, x_{f^{-1}(i)}\rangle\overline{\langle y, x_{f^{-1}(i)}\rangle} = \sum_{i\in I}\langle x, x_i\rangle\overline{\langle y, x_i\rangle} = \langle x, y\rangle$. Now assume that the set J of those elements which are left fixed by f is not empty. Denote by X the smallest closed subspace of H containing x_j for all $j \in J$. X is the smallest closed subspace containing $\{\langle x_j\rangle \mid j \in J\}$ and φ_f induces the identity on X. For the latter claim observe that φ_f induces the identity on the subspace spanned by $\{x_j \mid j \in J\}$ and X is the (topological) closure of that subspace. By continuity φ_f induces the identity on X too. Now, φ_f induces an ortholattice automorphism ρ_f on $Sub(H)$ such that for any $i \in I$, $\rho_f(\langle x_i\rangle) = \langle x_{f(i)}\rangle$. It is also evident that ρ_f induces the identical map on $[0, X]$. Thus the symmetry condition is verified.
For the other direction note that the existence of a basis guarantees that the lattice of propositions denoted by $Prop_{\mathcal{L}}$ has infinite height and observe that by Theorem 32 there exists an orthomodular space H and an isomorphism $\Phi : Prop_{\mathcal{L}} \to Sub(H)$. We now exploit the symmetry property of $\mathcal{L}$ to prove that H must be a classical (infinite-dimensional) Hilbert space. Let Δ be a basis for $\mathcal{L}$ with respect to which $\mathcal{L}$ is symmetric. Let $(\sigma_i)_{i\in I}$ be a corresponding family of pointers. We look at the family $\Phi([\sigma_i])_{i\in I}$. Put $\langle x_i\rangle = \Phi([\sigma_i])$. This is an infinite pairwise orthogonal family of one dimensional subspaces (rays) of H. We will construct a family $(y_i)_{i\in I}$ of pairwise orthogonal elements of H such that for any $i, j \in I$ we have $\langle y_i, y_i\rangle = \langle y_j, y_j\rangle$. Then it follows by Soler's theorem that H is a classical Hilbert space.

Let $i_0 \in I$ be fixed. Then for every $j \in I, j \neq i_0$ consider the permutation f_j of I defined as follows.

$$f_j(i_0) = j \text{ and } f_j(j) = i_0,\ f_j(i) = i \text{ else.}$$

The symmetry condition then guarantees that for every $j \in I, j \neq i_0$ there exists an automorhism φ_j of $Sub(H)$ such that

$$\varphi_j(\langle x_{i_0}\rangle) = \langle x_j\rangle$$

and, moreover, induces the identity on $[0, X]$, where X is the smallest closed subspace of H containing $\langle x_i \rangle$ for $i \neq i_0, j$. Clearly X has dimension greater than 2. In fact, it is infinite-dimensional. Mayet's theorem then yields that φ_j is induced by some unitary operator ρ_j of H. For $j \neq i_0$ put $y_j = \rho_j(x_{i_0})$ and $y_{i_0} = x_{i_0}$. Since ρ_j is unitary and the $\langle x_i \rangle$'s are pairwise orthogonal, the family $(y_j)_{j \in I}$ is as required in Solèr's theorem and H must be an infinite-dimensional classical Hilbert space.

We still need to prove that H presents $\mathcal{L}$. For this we first need to define the function Ψ. Define $\Psi : Fml \to Sub(H)$ by $\Psi(\alpha) = \Phi([\alpha])$. It is routinely verified that Ψ satisfies the conditions required.
We need to show

- 1. $\mathcal{C} = \mathcal{C}_{H,\Psi}$
- 2. If $\mid\!\sim = \vdash_x$, then for any α, $\mid\!\sim_\alpha = \mathcal{F}_{H,\Psi}(\alpha, \vdash_x)$

For 1. let $\mid\!\sim \in \mathcal{C}$ be given. We need to find a $\vdash_x \in \mathcal{C}_{H,\Psi}$ such that $\mid\!\sim = \vdash_x$. Let σ be a pointer to $\mid\!\sim$ and $\langle x \rangle = \Phi([\sigma]) = \Psi(\sigma)$. We have $\alpha \mid\!\sim \beta$ iff $\sigma \mid\!\sim \alpha \rightsquigarrow_s \beta$ iff $[\sigma] \subset [\alpha \rightsquigarrow_s \beta]$. This is equivalent to $\langle x \rangle \subset \Psi(\alpha \rightsquigarrow_s \beta)$ which says $\alpha \vdash_x \beta$. Thus $\mid\!\sim = \vdash_x$. For a given $\vdash_x$ the same reasoning applies to find a $\mid\!\sim \in \mathcal{C}$ such that $\vdash_x = \mid\!\sim$.
For 2. let $\mid\!\sim = \vdash_x$. Note that $\beta \mid\!\sim_\alpha \gamma$ iff $\alpha \mid\!\sim (\beta \rightsquigarrow_s \gamma)$ iff $\alpha \vdash_x (\beta \rightsquigarrow_s \gamma)$ iff $\beta \vdash_{Ax} \gamma$. But $\vdash_{Ax} = \mathcal{F}_{H,\Psi}(\alpha, \vdash_x)$. ▮

Remark: The reader may have noticed that in he above proof we did not use the second condition of the definition of a basis. In fact the argument works without that condition. If we omit the second condition we can no longer say that Δ contains 'the whole information' of $\mathcal{L}$. Instead, its intuitive function would be to guarantee that $\mathcal{L}$ is 'rich in information' in that it contains infinitely many non orthogonal consequence relations, which thus are not encoded in each other.[1]

10.3.3 Reflecting on the Representation Theorem

Let us now reflect on the intuitive meaning the technical result we called the representation theorem might have. In a sense this book describes a journey. We

[1]The authors cannot refrain from putting themselves in a mystic's boots for a moment. What's the intuitive content of the combination of holicity and symmetry which obviously is at the heart of the logics presentable by infinite-dimensional Hilbert spaces? A mystic might say that these properties represent the inherent 'unity' of a Hilbert space logic. Given a holistic logic $\mathcal{L}$, let Δ be a basis for $\mathcal{L}$ and let $\mathcal{L}$ be symmetric with respect to Δ. Then we know that every consequence relation $\mid\!\sim_0$ of $\mathcal{L}$ is encoded in some member $\mid\!\sim_1$ of Δ via the internalising connective and vice versa. So, $\mid\!\sim_0$ and $\mid\!\sim_1$ aren't essentially different. This is holicity. But what about different elements $\mid\!\sim_0, \mid\!\sim_1 \in \Delta$? Aren't they essentially different? Well, the mystic might say, symmetry expresses a sort of indistinguishability of the basic consequence relations, the sort of indistinguishability we encounter so often at the level of quantum mechanics, for instance in connection with the symmetry of the wave function of many particle systems. This time, the symmetry is at the logical level. So, the mystic might say, there is some 'hidden unity' behind the apparent diversity and variety of the consequence relations of a Hilbert space logic.

said at the beginning that we would aim at finding logical structures in Hilbert space. This is in full accordance with with what Birkhoff and von Neumann had in mind. Our first deviation from Birkhoff and von Neumann was that in our treatment of propositions we focused on projections in Hilbert space rather closed subspaces. This led us to what we called the dynamic viewpoint. We elaborated on this technicaly by introducing and studying structures we called M-algebras, which are abstractions from the lattice of projections in Hilbert space.

Our second deviation from Birkhoff and von Neumann is what we called the local viewpoint. In the Birkhoff-von Neumann paper as well as in our study of M-algebras the concept of a state was a a primitive notion. We then set out to enquire into the logical nature of the concept of a physical state. Finally, we were led to the concept of a holistic logic. These structures constitute a sort of synthesis of both viewpoints. The dynamic nature of propositions is transparent in these structures and states are represented as precisely defined logical entities. Moreover the framework of holistic logics turns out to be a precise framework in which certain natural vague intuitions arising in connection with he phenomena described in Chapter 8 such as wholeness, interconnectedness etc. have a precise meaning. In a sense, essential features of quantum reality are mirrored on this platform in a surprising way.

We found that the natural vehicles for presenting holistic logics are certain vector spaces called orthomodular spaces, in particular Hilbert spaces. We then asked the question how to characterise those holistic logics that are presentable by a Hilbert space. We gave a positive answer to this question in the Representation Theorem. It is remarkable that the crucial condition in this characterisation is a symmetry condition.

Hilbert spaces, i.e. the structures constituting the core of the formalism of quantum mechanics, are special orthomodular spaces and thus present holistic logics. We called the logics presentable by Hilbert spaces Hilbert space logics. In the above representation theorem we characterised Hilbert space logics among holistic logics. The crucial condition in that characterisation was a symmetry property. This is remarkable. It means that not only does symmetry play an enormous role inside the formalism of quantum mechanics but that it is also at the heart of the formalism itself.

Let us reflect at this point. What was the course of the experiment this book is about so far? Where did the journey on which we embarked lead us so far?

Let us summarise: At the beginning of the journey there was an epistemological issue, namely the issue of reality in quantum mechanics. We then tried to give this philosophical problem a scientific twist by translating it into a problem of logic. The problem was: Construct logical monads! Our intuition was that these logical monads should be constructed as keeping their commitment to the structure of reality at a minimum. Our aim was to construct logical systems which constitute their own semantic structures.

This was the starting point of the journey. We arrived at the concept of logical structures which we called holistic logics. Their salient properties are self-referential soundness and completeness and the properties expressed in the

no windows theorems and the limiting case theorem. These properties reflect their monadic nature so to speak.

We then asked the question how these structures can be presented and in searching for an answer we hit upon the concept of an orthomodular space and, in particular, the concept of a Hilbert space. So our journey so far led us from the intuition of logical monadology to Hilbert space, which constitutes the core of the mathematical formalism of quantum mechanics.

10.4 Formal Reflections on the Connectives in Hilbert Space Logics

This Section is still incomplete in this draft. But it is also not necessary for understanding the subsequent chapters

Quantum logic, however it may be defined, is certainly one of those branches of logic in which the connectives are least understood. In this section we take a closer look at the problem of the connectives in quantum logic. This is another application of the 'local viewpoint'. Given a Hilbert space H and some $x \in H$ we may look at the consequence relation or, more generally, the inference operation corresponding to x for a language without connectives, i.e just a set of atomic propositions. We may then look at various languages containing connectives, and we can study conservative extensions of the inference operation to these languages. In this section, which is of a purely technical nature, we study such extensions.

In this we combine two approaches, namely the approach to quantum logic taken by Engesser and Gabbay in [14] and the approach to introducing and studying connectives in a general logical setting taken by Lehmann in [40] and [41]. The approach adopted in [**?**] permits us to start with a language without connectives. In fact, several interesting properties of the inference operations induced by quantum states can be studied in the absence of connectives. This is in contrast to approaches in the spirit of the Birkhoff and von Neumann paper [2] which start with the connectives as presented by the lattice operations of the lattice of closed subspaces of a Hilbert space.

We start with a quantum inference operation as introduced and studied in [40] in a poor language without connectives as described. We then study possible conservative extensions to richer languages containing connectives. We will investigate several such extensions and discuss their properties from various points of view.

10.4.1 Quantum Consequence Relations and Inference Operations

Given a non-empty set $\mathcal{P}$ which we regard as a set of atomic propositions. Thus the language with which we start has no connectives. Given a Hilbert space H, an element x of H and a function $\Psi : \mathcal{P} \to Sub(H)$, where $Sub(H)$ denotes the

set of all closed subspaces of H. As usual we write A for $\Psi(\alpha)$ as well as for the projector corresponding to $\Psi(\alpha)$. We then have the consequence relation $\vdash_x$ over $\mathcal{P}$ presented by Ψ:

$$\alpha \vdash_x \beta \text{ iff } Ax \in B$$

As observed by Lehmann, this definition can be extended to the definition of an (infinitary) inference operation as follows. Let $\mathcal{A}$ be any set of (atomic) formulas. We may then define what it means to say that β is a consequence of $\mathcal{A}$. Namely, consider $A =: \bigcap\{\Psi(\alpha) \mid \alpha \in \mathcal{A}\}$. Note that that A is again a closed subspace of H so that we may define β to be a consequence of $\mathcal{A}$, i.e. $\beta \in C(\mathcal{A})$, iff $Ax \in B$.
Denote the set of x-consequences of $\mathcal{A}$ by $\mathcal{C}_x(\mathcal{A})$. Context permitting we omit the subscript x. These inference operations called quantum inference operations have certain nice properties. They for instance satisfy the following conditions as is routinely checked.

$$\text{For any } \mathcal{A} \subset \mathcal{P} \text{ we have } \mathcal{A} \subset C(\mathcal{A}) \ \textit{Inclusion}$$

and

$$\mathcal{A} \subset \mathcal{B} \subset C(\mathcal{A}) \text{ implies } C(\mathcal{A}) = \mathcal{C}(\mathcal{B}) \ \textit{Cumulativity}$$

In [40] inference operations satisfying the above conditions are called C-logics. C-logics admit a particularly smooth characterisation in terms of a representation theorem (see [40]) and are worth studying in their own right. Keep in mind that quantum consequence operations are C-logics.

Let us now consider the closure $\mathcal{L}$ of $\mathcal{P}$ under the unary connective $\neg$ and the binary connective $\wedge$, i.e. the smallest language $\mathcal{L}$ containing $\mathcal{P}$ and closed under $\neg$ and $\wedge$ and let us consider extensions of Ψ to $\mathcal{L}$. We shall now consider conservative extensions of quantum consequence relations and inference operations to the richer language $\mathcal{L}$.

Lehmann considers three conditions the connectives $\wedge$ and $\neg$ should satisy.

$\wedge$-R $\mathcal{C}(\mathcal{A}, \alpha \wedge \beta) = \mathcal{C}(\mathcal{A}, \alpha, \beta)$

$\neg$ R1 $\mathcal{C}(A, \alpha, \neg\alpha) = \mathcal{L}$

$\neg$ R2 if $\mathcal{C}(A, \neg\alpha) = \mathcal{L}$, then $\alpha \in \mathcal{C}(A)$.

Call a set $\mathcal{A}$ of formulas a theory if $\mathcal{C}(\mathcal{A}) = \mathcal{A}$. Call $\mathcal{A}$ consistent if $\mathcal{C}(\mathcal{A})$ is not the full language.

10.4.2 The Birkhoff-von Neumann Extension

We first consider the following extension of Ψ to $\mathcal{L}$.

$$\Psi(\alpha \wedge \beta) = \Psi(\alpha) \cap \Psi(\beta)$$

$\Psi(\neg\alpha) = \Psi(\alpha)^{\perp}$, where $\Psi(\alpha)$ alpha denotes the orthogonal complement of $\Psi(\alpha)$.

Note that Ψ goes into $Sub(H)$ and and the extended consequence relation and inference operation is defined analogously to the case above by (1).

This extension is in the spirit of [2] in that we invoke orthogonal complement formation in Hilbert space in order to present negation. We refer to it as the Birkhoff-von Neumann extension . Recall that have proved that these consequence relations have the Sasaki hook as an internalising connective.

10.4.3 The Lehmann Extension

In [40] Lehmann proved the following remarkable result.

Theorem 10.7 (Lehmann) *Let $\mathcal{C}$ be a C-logic over $\mathcal{P}$. Then there exists a C-logic $\mathcal{C}'$ over $\mathcal{L}$ that satisfies $\wedge - R$, $\neg R1$, $\neg R2$, such that, for any $\mathcal{A} \subset \mathcal{P}$, $\mathcal{C}(\mathcal{A}) = \mathcal{P} \cap \mathcal{C}'(\mathcal{A})$*

Since the inference operations $\mathcal{C}_x$ are $\mathcal{C}$-logics, it follows from the above theorem that they admit a conservative extension as described in the theorem.

In [40] an explicit construction for such a conservative extension is given. We now describe the Lehmann extension for quantum consequence operations. Given $y \in H$, put $T_y =: \{A \in Sub(H) \mid y \in A\}$.

The following result characterises the consistent theories of a quantum consequence operation.

Theorem 10.8 *Let $\mathcal{A} \subset Sub(H)$, $x \in H$. Then $\mathcal{A}$ is an x-consistent theory iff there is a y not orthogonal to x such that $\mathcal{A} = T_y$.*

Proof. For the direction from left to right put $A =: \bigcap \mathcal{A}$ and $Ax =: y$. Then $\langle y \rangle \in \mathcal{C}(\mathcal{A})$. Since $\mathcal{C}(\mathcal{A}) = \mathcal{A}$, we have $\langle y \rangle \in \mathcal{A}$. Thus $A \subset \langle y \rangle$. Since A is assumed to be consistent and $\langle y \rangle$ is one-dimensional, we have $A = \langle y \rangle$. It follows that $\mathcal{A} \subset \mathrm{T}_y$.

Now assume assume $B \in T_y$. Then $Ax \in B$. This says that $B \in \mathcal{C}(\mathcal{A})$. Again, since $(\mathcal{A}) = \mathcal{A}$, we have $B \in \mathcal{A}$. It follows that $T_y \subset \mathcal{A}$. We have now established that $\mathcal{A} = T_y$.

It remains to be shown that y is not orthogonal to x. Suppose to the contrary that $y \in \langle x \rangle^{\perp}$, i.e. $Ax \in \langle x \rangle^{\perp}$. This means that $Ax = 0$ and thus $\mathcal{A}$ is inconsistent contrary to the hypothesis.
For the other direction it suffices to show that $\mathcal{C}(T_y) \subset T_y$. Let $B \in \mathcal{C}(T_y)$. It follows that the operator corresponding to $\langle y \rangle$ applied to x is in B. We then have $\langle y \rangle \subset B$ and thus $B \in T_y$.
∎

Given a quantum inference operation $\mathcal{C}_x$ over $\mathcal{P}$ presented by the function $\Psi : \mathcal{P} \to Sub(H)$. Consider $H_x := (\langle x \rangle^{\perp})^c$. and define the function $\Psi_L : \mathcal{P} \to 2^{H_x}$ presenting the Lehmann extension as follows. For $\alpha \in \mathcal{P}$ define

$$\Psi_L(\alpha) = \Psi(\alpha) \cap H_x,$$

For the connectives define

$$\Psi_L(\neg\alpha) = (\Psi_L(\alpha))^c,$$

where $(\Psi_L(\alpha))^c$ is the complement of $\Psi(\alpha)$ in H_x.

$$\Psi_L(\alpha \wedge \beta) = \Psi_L(\alpha) \cap \Psi_L(\beta)$$

Let A be any subset of H_x. Then we denote by A^* the smallest closed subspace of H containing A and as always, if there is no danger of confusion, the corresponding projection operator. For $A \in Sub(H)$ we put $A^x = (A \cap H_x)^*$. We can now define the Lehmann extension $\mathcal{C}^L$ omitting the subscript x.

Definition 10.6 *Let $\mathcal{C}_x$ be a quantum inference operation over $\mathcal{P}$ presented by the function Ψ. Let $\mathcal{A}$ be any set of formulas of $\mathcal{L}$. Then define $\mathcal{C}^L(\mathcal{A})$ as follows. Let β be any formula of $\mathcal{L}$. Then consider $S =: \bigcap\{\Psi^L(\alpha) \mid \alpha \in \mathcal{A}\}$. Now, if $S^*x \in S$, we say that $\beta \in \mathcal{C}^L(\mathcal{A})$ if $S^*x \in \Psi^L(\beta)$. If not $S^*x \in S$ we define $\beta \in \mathcal{C}^L(\mathcal{A})$ if $S \subset \Psi^L(\beta)$.*

This definition is the result of applying Lehmann's general construction as given in the proof of Theorem 1 in [40] to what he there calls quantum logics. te that whenever we have $\alpha \vdash_x \perp$, we must have the second case in the above definition, i.e. $\Psi'(\alpha)$ must be empty which in the atomic case means $\Psi(\alpha) \subset \langle x \rangle^\perp$ as required for conservativity.

Lemma 10.7 *Let $A \in Sub(H)$ and $x \in H$. Then $Ax \in \langle x \rangle^\perp$ implies $x = 0$.*

Proof. Assume $Ax \in \langle x \rangle^\perp$. This means that $\langle Ax, x \rangle = 0$. Recall that $AA = A$. It follows that $\langle AAx, x \rangle = 0$. Since A is self-adjoint, we get $\langle Ax, Ax \rangle = 0$ and thus $Ax = 0$. ▮

Lemma 10.8 *Let $A \subset Sub(H)$ and suppose $Ax \neq 0$. Then $Ax = A^x x$.*

Proof. First observe that $A^x \subset A$. Moreover, note that, if $Ax \neq 0$, then we have by the last lemma that not $Ax \in \langle x \rangle^\perp$ and thus $Ax \in A \cap H_x$, i.e. $Ax \in A^x$. Since Ax and $A^x x$ are the unique elements of A and A^x respectively which are closest to x, it follows that $Ax = A^x x$. ▮

Lemma 10.9 *Let $S \subset Sub(H)$ and let $A = \bigcap\{B \cap H_x \mid B \in S\}$. Then $A^* = (\bigcap\{B \mid B \in S\})^x$.*

Theorem 10.9 *The Lehmann extension is a C-Logic and a conservative extension of $\mathcal{C}_x$ such that*

- *(i) $\vdash_x \alpha \wedge \beta$ iff $\vdash_x \alpha$ and $\vdash_x \beta$.*
- *(ii) $\vdash_x \neg\alpha$ iff not $\vdash_x \alpha$.*

Proof. Let us verify that the Lehmann extension is in fact a C-logic. So given sets of formulas $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}^{\mathcal{L}}(\mathcal{A})$. We then need to prove that $\mathcal{C}^L(\mathcal{A}) = \mathcal{C}^L(\mathcal{B})$. Let $S = \bigcap\{\Psi^L(\alpha \mid \alpha \in \mathcal{A}\}$ and $T = \bigcap\{\Psi^L(\beta) \mid \beta \in \mathcal{B}\}$. We have to consider four cases.
Case 1: $S^*x \in S$ and $T^*x \in T$.
Case 2: $S^*x \in S$, but not $T^*x \in T$.
Case 3: not $S^*x \in S$, but $T^*x \in T$.
Case 4: not $S^*x \in S$ and not $T^*x \in T$.
We prove case 1. Note that $T \subset S$. We have for every $\beta \in \mathcal{B}$ that $S^*x \in \Psi^L(\beta)$. It follows that $S^*x \in T$ and thus $S^*x = T^*x$. Now let $\beta \in \mathcal{C}^L(\mathcal{B})$. This means $T^*x \in \Psi^L(\beta)$ and thus $S^*x \in \Psi^L(\beta)$ which means that $\beta \in \mathcal{C}^L(\mathcal{A})$.
We prove case 2. First note that $S^*x \in \Psi(\beta)$ for all $\beta \in \mathcal{B}$ anf thus $S^*x \in T$. Let $\beta \in \mathcal{C}^L(\mathcal{B})$. Then we have $T \subset \Psi^L(\beta)$. It follows that $\S^*x \in \Psi^L(\beta)$.
We prove case 3. In this case $\mathcal{B} \subset \mathrm{C}^L(\mathcal{A})$ implies $S \subset T$. Since $T \subset S$, we have $S = T$. If $\beta \in \mathrm{C}^L(\mathcal{B})$, we have $T^*x = S^*x \in \Psi(\beta)$, hence $\beta \in \mathcal{C}^L(\mathcal{A})$.
We prove case 4. In this case we, again, have $S = T$. $\beta \in \mathcal{C}^L((\mathcal{B})$ says $T \subset \Psi^L(\beta)$. Thus $S \subset \Psi^L(\beta)$, which means $\beta \in \mathcal{C}^L(\mathcal{A}$.

We now prove that the Lehmann extension is conservative. For this we need to show that for any set $\mathcal{A}$ of formulas of $\mathcal{P}$ and any $\beta \in \mathcal{P}$ we have $\beta \in \mathcal{C}^L(\mathcal{A})$ iff $\beta \in \mathcal{C}_x(\mathcal{A})$. Assume that $\mathcal{C}_x$ is presented by Ψ. Assume $\beta \in \mathcal{C}^L(\mathcal{A})$. Let $S = \bigcap\{\Psi^L(\alpha) \mid \alpha \in \mathcal{A}\}$. Let $A = \bigcap\{\Psi(\alpha) \mid \alpha \in \mathcal{A}\}$. Assume the first case in the definition 10.6, namely that $S^*x \in S$. It follows that $S^*x \neq 0$. Thus $\beta \in \mathcal{C}^L(\mathcal{A})$ iff $S^*x \in \Psi^L(\beta)$. Since, by lemma 10.8 we have $S^* = A^x$, we have $A^x x \in \Psi^L(\beta)$. Noting that $\Psi^L(\beta) \subset \Psi(\beta)$ we getthat $Ax \in \Psi(\beta)$, which says that $\beta \in \mathcal{C}_x(\mathcal{A}$. Now assume the second case of definition 10.6, i.e. not $S^*x \in S$. This means that $S^*x \in \langle x\rangle^\perp$. This is, by lemma 10.7 possible only if $S^x x = 0$ and thus, by lemma **??** $Ax = 0$. But this says $\beta \in \mathcal{C}_x(\mathcal{A})$.
The other direction is established by similar reasoning. Suppose $\beta \in \mathcal{C}_x(\mathcal{A}$. This says that $Ax \in \Psi(\beta)$. Assume $Ax \neq 0$. It follows by lemma 10.8 and lemma **??** that $A^x x = Ax = S^*x \in S$. Thus $S^*x \in \Psi^L(\beta)$. This is the first case in Definition 10.6 and we have $\beta \in \mathcal{C}^L(\mathcal{A})$. Now suppose $Ax = 0$. It follows that S is empty. So $S \subset \Psi^L(\beta)$. We have the second case in Definition 10.6 and $\beta \in \mathcal{C}^L(\mathcal{A})$.
∎

10.4.4 The Engesser-Gabbay Extension

The following extension is very much in the spirit of [14].

Definition 10.7 *Given a set of atomic formulas (this is our object language, it has no connectives). Let $\Box$ be a binary connective, $\neg$ a unary connective. The intended meaning of $\Box(\alpha, \beta)$ is "β is provable from α", the intended meaning of $\neg$ is classical negation at the metalevel. Define the language ML by the following clauses.*

- *(i) If $\alpha, \beta \in \mathcal{P}$, then $\Box(\alpha, \beta) \in ML$.*

- (ii) *If* $\varphi, \psi \in \mathcal{P} \cup ML$*, then* $\Box(\varphi, \psi) \in ML$.
- (ii) *If* $\varphi, \psi \in ML$*, then* $\neg\varphi, \varphi \wedge \psi \in Ml$.

Now consider the language $\mathcal{L}^* = \mathcal{P} \cup ML$. Note that the Boolean connectives $\neg$ and $\wedge$ act on metastatements only. Again, given $\Psi : \mathcal{P} :\rightarrow Sub(H)$ presenting the quantum consequence relation $\vdash_x$. We extend Ψ to a function (again called Ψ) $\Psi : \mathcal{L}^* \rightarrow 2^H$. In fact it goes into $Sub(H)$. In this we shall make use of orthogonal complement formation. But, as will be seen, this does not mean that orthogonal complement formation in any way 'corresponds' to a connective (negation). Rather it plays an auxiliary role role in the mathematical definition of the proof operator.

We define the extension of Ψ as follows:

$$\Psi(\Box(\varphi, \psi) = (\langle x \rangle)^\perp + (\langle x \rangle \cap (\Psi(\varphi)^\perp + (\Psi(\varphi) \cap \Psi(\psi)))^\perp$$

$$\Psi(\neg\varphi) = \Psi(\varphi)^\perp$$

We now define the extension of $\vdash_x$, again so denoted, as usual:
For any $\varphi, \psi \in \mathcal{L}^*$, $\varphi \vdash_x \psi$ iff: $\Psi(\varphi)x \in \Psi(\psi)$. And we claim that this extension is a conservative extension of $\vdash_x$ over $\mathcal{L}^*$ which is self-referentially complete.

We now define the notion of truth for ML in a straightforward way. We then call a consequence relation $\mid\!\sim$ self-referentially complete iff for all $\varphi \in ML$ we have $\mid\!\sim \varphi$ iff φ is true.

Definition 10.8 *Let* $\mid\!\sim$ *be any consequence relation and* ML *the language just defined. We say that*

- (i) *If* $\alpha, \beta \in \mathcal{P}$*,* $\Box(\alpha, \beta$ *is true iff* $\alpha \mid\!\sim \beta$
- (ii) *analogously for clause* (ii) *in the last definition.*
- $(iii($ *If* $\varphi = \neg\psi \in ML$*, then* φ *is true iff* ψ *is not true and* $\varphi \wedge \psi$ *is true iff* φ *is true and* ψ *is true.*

10.4.5 Discussing Negation

Let us now compare the three types of negation introduced in the last section. The Birkhoff-von Neumann extension is undubtedly the richest extension. However, the negation presented by orthogonal complement formation admits no plausible intuitive interpretation. Moreover, these consequence relations have another undesirable property. Namely, they areare 'classically inconsistent' in that for any such quantum inference operation $\mathcal{C}$ there exists a classical contradiction α such that $\alpha \in \mathcal{C}(\top)$. On other hand, clearly the Birkhoff-von Neumann extensions do have some goodies. They admit a internalising connective, namely the Sasaki hook $\leadsto_s$, which is definable by $\neg$ and $\wedge$. They are self-referentially complete and, again, the proof operator is definable by $\neg$ and $\wedge$. The Lehmann extension is classically consistent. Yet it does not admit a definable internalising

connective and it does not admit a definable proof operator. It thus is no quantum logic, i.e. it cannot be presented by a Hilbert space in in the sense of [3]. The Engesser-Gabbay extension is classically consistent and is sef-relferentially complete. The connectives $\neg$ and $\wedge$ behave absolutely clasically. All connectives admit a natural interpretation.

Proposition 10.4 *The Birkhoff-von Neumann extension is classically inconsistent. But it admits an internalising connective and a proof operator both definable by $\neg$ and $\wedge$. Negation does not satisfy $\neg$-R2.*

Proposition 10.5 *The Lehmann extension is classically consistent. Negation satisfies $\neg$-R2. It admits no internalising connective and no proof operator definable by $\neg$ and $\wedge$.*

The following corollary answers a question raised by Lehmann in [**?**], namely the question whether any $\mathcal{C}$-logic is presentable by a Hilbert space, i.e. that it is a quantum logic. The answer is negative.

Corollary 10.1 *The Lehmann extension is no quantum logic.*

Proof. We can assign a truth value $V(\alpha)$ to every $\alpha \in \mathcal{L}$ in a natural way as follows. For atomic $\alpha \in \mathcal{P}$ define $V(\alpha) = 1$ iff $\vdash_x \alpha$ and V and extend V canonically to $\mathcal{L}$. Then we see, using Lemma 1, that we have $V(\alpha) = 1$ iff $\vdash_x \alpha$. This means that any α such that $\vdash_x \alpha$ is classically consistent. That the Lehmann extension satisfies $\neg - R2$ follows from theorem 1. It is, however, interesting to verify this in the special case of quantum inference operations. So let $\alpha \wedge \neg\beta \vdash_x \perp$.

Let us now see why the Lehmann extension does not admit an internalising connective definable by $\wedge$ and $\neg$. Intuitively, the reason for this is that because of lemma 1 the internalising connective, if existing, would have to behave 'truth-functionally', whereas the consequence relation over $\mathcal{P}$ does not behave so. i.e. whether for $\alpha, \beta \in \mathcal{P}$ we have $\alpha \vdash_x \beta$ does not depend on the provability or unprovability of α and β alone. So assume there is an internalising connective $C(\alpha, \beta)$ definable by $\wedge$ and $\neg$. Then, for any $\alpha, \beta \in \mathcal{P}$ we would have that $\alpha \vdash_x \beta$ iff $\vdash_x C(\alpha, \beta)$. This means, in particular, that whether $\alpha \vdash_x \beta$ holds defends only on the 'truth values' $V(\alpha)$ and $V(\beta)$. Now, let α, β, γ such that not $\vdash_x \alpha$, not $\vdash_x \gamma$ and $\alpha \vdash_x \beta$ and not $\gamma \vdash_x \beta$. This constellation is perfectly possible. Now, since $C(\alpha, \beta)$ is an internalising connective, $\alpha \vdash_x \beta$ would imply $\vdash_x C(\alpha, \beta)$ from which in turn it would follow that $\gamma \vdash_x \beta$, since $V(\alpha) = V(\gamma)$. But this is a contradiction.

∎

Proposition 10.6 *The Engesser-Gabbay extension is classically consistent. Negation satisfies $\neg$-R2.*

10.4.6 Comment on $\neg - R2$

We look at $\neg - R2$ from the point of view of Hilbert space logics. What does $\neg_R 2$ mean in terms of CRS of classical logic? Given the classical consequence relation $\vdash$. Then $\alpha \wedge \neg\beta \vdash \perp$ says that if we revise $\vdash$, then the revised consequence relation $\vdash_{\alpha \wedge \neg\beta}$ proves β. Since in the case of classical logic revision by $\alpha \wedge \neg\beta$ amounts to first revising by α and subsequently by $\neg\beta$ or the other way round, we may interpret $\neg - R2$ as follows. It says in particular: if revision by α yields a consequence relation with which β is inconsistent, then β is a consequence of α in the original consequence relation. $\vdash \neg\alpha$ says $\alpha \vdash \perp$. In the Engesser-Gabbay extension this can be retrieved, namely we have $\vdash \neg\alpha$ iff $\vdash \Box(\alpha, \perp)$. So there is no loss of information in the Engesser-Gabbay extension as compared to the Birkhoff-von Neumann extension.

Chapter 11

Some Speculative Reflections

11.1 A Look at the Measurement Problem

In this chapter we use the concept of a tensor product and the fact that the combination of two physical systems is the tensor product of their respective Hilbert spaces. In this draft we have not yet introduced the concept of a tensor product of Hilbert spaces. The reasons for this are those mentioned in Chapter 4. In the final version this will of course be done.

11.1.1 General Remarks

In their article on quantum logic in the Handbook of Philosophical Logic, Dalla Chiara and Giuntini arrive at the conclusion that traditional quantum logic hasn't made a significant contribution to the solution of the various (foundational) puzzles with which quantum mechanics confronts us. In particular, there is no satisfactory account yet for the puzzle which has become known as the measurement problem. It is not just quantum logic that has no solution to offer but there is no approach whatsoever that can account for this puzzle in a universally accepted way. In this chapter we investigate what resources we have in our framework to attack this problem.

Before describing the problem in detail in the dramatic form of Schroedinger's cat let us explain its general nature and why it is crucial for the understanding of quantum mechanics and in particular its mathematical formalism. In quantum mechanics the concept of measurement plays a vital role - in contrast to classical mechanics. Recall that it is one of the basic principles of QM that, given an observable A represented mathematically as the Hermitian operator (again called) A, then the eigenvalues of A are those quantities that we can find as values of observable A when a *measurement* of A is performed. The uncertainty relations are statement concerning 'non-simultaneous measurability'

of certain observables. Generally, QM makes statements about the outcome of measurements.

Now, the process of measurement is a physical process itself and we may expect the formalism of QM to give us an adequate description of the process of measurement itself. This, however, is not the case. Essentially, the mesurement problem or the measurement paradox as it is, perhaps more adequately, called sometimes consists in the fact that on the hand the formalism of quantum mechanics is about measurements but on the other hand seems to give incorrect results when applied to the process of measurement itself.

Generally, what are the characteristics of measurement in physics? Measurement is a physical process involving the interaction of two systems, the system to be measured and the measuring system, also called the measuring instrument. The systems interact in such a way that one of the interacting systems, namely the measururement instrument, 'gives' us the value of a certain observable pertaining to the system to be measured. This applies equally to quantum physics and classical physics. It is, however, clear that not every interaction between two systems constitutes a measurement. There are of course physical interactions between systems in which neither of them 'measures' the other. Rather, in practice, the measuring instrument is a macroscopic object with a scale or a screen from which the result of measurement can be read off by the human eye. Bohr famously insisted that the entire experimental arrangement even in the case of quantum measurement must be describable in terms of classical physics. Landau and Lifschitz, in their classic textbook, say that quantum mechanics 'presupposes' classical mechanics.

In principle, the formalism of QM permits us to treat any interaction of two systems. Both are, in the formalism, represented by their corresponding Hilbert spaces the tensor product of which represents the composite system. But how is the particular nature of the measurement process reflected in the formalism? In particular, how are the different roles of the system to be measured and the measuring instrument reflected in the formalism? What, if anything, is special about the Hilbert spaces of measuring instruments? As far as we see, these questions have no answer within the formalism of QM. All one can do in this framework is to represent both systems as (in general infinite dimensional) Hilbert spaces and apply the mathematical machimery of Hilbert space tensor products. In this sense that we may say that not only can the formalism of quantum mechanics not *treat* the process of measurement correctly but that it cannot even *define* it.

The issue of measurement in quantum mechanics is closely linked to another issue, namely that of the 'collapse of the wave function' or (synonymously) the projection postulate. Recall that the projection postulate says the following. Assume a measurement of an observable A represented by an Hermitian operator (denoted again by) A is performed. Then after measurement the system is in an eigenstate of A and the corresponding eigenvalue is the value of A measured. It is important to note this link between measurement, which is a physical process, and the phenomenon of 'collapse'. It is not just that we experience the 'strange phenomenon' of collapse (projection) in quantum mechanics but we have to

bear in mind that it is in the process of measurement that it occurs. We may therefore expect a theory of measurement to *explain* this phenomenon rather than to presuppose it.

Our framework of holistic logics provides us with a refined look at this. Methodologically, we may view this chapter as 'playing the game of measurement' in a logical framework, namely the framework of holistic logics in analogy to the way we 'played the game of reality' in previous chapters.

11.1.2 The measurement problem in a nutshell

We view the measurement problem in the dramatic version of Schroedinger's cat. Given a quantum system, say an electron. We want to measure the z-component of the spin of the electron. It is known that this observable can assume only two values: "up" and "down". We assume that the experimental arrangement of the measuring process is as follows. In the case "spin = down" some device is triggered which kills a cat. If we have "spin = up" the cat stays alive. The cat is thus the measuring instrument.

Assume that before measurement the electron is (as a fermion) in the singlett state

$$(1)\ 1/\sqrt(2)(\mid up > - \mid down >)$$

Call the system consisting both of the cat, the measuring instrument, and the electron, the system to be measured, the composite system. Then the formalism of quantum mechanics tells us that, after measurement, the composite system is in the following state:

$$(2)\ 1/\sqrt(2)(\mid alive >\mid up > - \mid dead >\mid down >)$$

So, after measurement the composite system is in a superposition. This is at odds with the facts. The facts are that after measurement the cat is either alive and spin up or the cat is dead and spin down, both with probability 1/2. The formalism, however, says that the z-component of spin is not sharp nor is the cat's life. This is what in the literature is often referred to as: "The cat is half alive and half dead". Such a state has never been observed. Rather the system is either in state

$$\mid alive >\mid up >$$

or in state

$$\mid dead >\mid down >$$

Both states have probability 1/2.

Therefore, the formalism of quantum mechanics does not provide the correct prediction when applied to the measurement process itself. This is the measurement problem in a nutshell. There seems to be no generally acknowledged solution to this problem yet.

11.1.3 Some more thoughts on measurement

We said that measurement involves two interacting systems and it is thus natural to represent the combination of these two systems as the combination of two holistic logics. In fact, we will confine ourselves to the case of those holistic logics that 'come' from a Hilbert space, i.e. Hilbert space logics. In this, however, we have to consider a feature of the process we haven't considered yet, namely that of *correlation.* Assume we want to measure an observable A pertaining to a certain system and assume A can adopt a family of values $(\lambda_i)_{i\in I}$. For this purpose we let it interact with a measuring instrument in such a way that after measurement the value λ_i assumed by observable A can be read off from a scale or a screen. We may thus view λ_i as the value of an observable pertaining to the measuring instrument. This observable is normally called the pointer observable. So, measuring A means *correlating* it with the pointer observable of the measuring instrument. And the values of the pointer observable can be read off from a scale or a screen. We have to reflect this notion of *correlation* of observables in our treatment of the measurement problem.

We said that in classical physics measurement doesn't pose a problem. Intuitively, in classical physics measurement is just 'looking' at the system to be measured and the state of that system is not changed in measurement. In our treatment of measurement this trivial feature of measurement in classical physics will have to be reflected as a limiting case similar to the way the logic of classical mechanics appears as a limiting case in the framework of holistic logics as made precise by the Limiting Case Theorem.

To summarise, in our 'playing the game of measurement' in logic we will have to reflect the following characteristics of the process of measurement:

- 1) Combining two systems
- 2) Correlating two systems
- 3)The 'classical' nature of the measuring system (instrument)
- 4)The temporal evolution in the process of measurement
- 5)The projection postulate ("collapse of the wave function")
- 5) Classical measurement as a limiting case

11.1.4 Combining and correlating Hilbert space logics

Given two Hilbert space logics $\mathcal{L}_\infty$, $\mathcal{L}_2$ with languages Fml_1 and Fml_2 respectively. We introduce the connective $\otimes$ to form formulas of the 'combined' language. If $\alpha \in Fml_1$ and $\beta \in Fml_2$, then $\alpha \otimes \beta$ is a formula of the combined language. We will use the symbol $\otimes$ both as denoting this connective and the algebraic operation 'tensor' for vectors and also for the operation of combining consequence relations. A word of caution is in order here. The reader is advised not to think of $\alpha \otimes \beta$ as saying something like 'α and β'. Rather, he may think

$\alpha \otimes \beta$ as making sense only in connection with the combined system. He may, intuitively, think of the combined language as talking about the 'whole', i.e. the combined system, and view the connective $\otimes$ as the 'connective of wholeness'.

Definition 11.1 *Given a Hilbert space logic $\mathcal{L}$, x_1, x_2.. a sequence of mutually orthogonal vectors which may contain the zero vector. Assume $y \in H$ is not orthogonal to all non-zero members of this sequence. Then we say that $\vdash_y$ is a superposition of $\vdash_{x_1}, \vdash_{x_2}, \ldots$.*

We will in the sequel, for the sake of brevity, use the term 'state' familiar from quantum mechanics also for denoting consequence relations of a Hilbert space logic.

Intuitively, we may look at the concept of a superposition in various ways. First, superpositions may be viewed as 'encoding' all its (consistent) components or 'containing' all the information of its (non-zero) components, since it is non-orthogonal to each of its (non-zero) components. Second, observe a superposition can be revised so as to yield any of its non-zero components or, put differently, superpositions can in principle 'collapse' into each of its non-zero components because a superposition is non-orthogonal to any of its non-zero components.

Definition 11.2 *Let $\mathcal{L}_1 = \mathcal{L}_{H_1,\Psi_1}$, and $\mathcal{L}_2 = \mathcal{L}_{H_2,\Psi_2}$ be Hilbert space logics. We define the combination $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2 =: \mathcal{L}_{H_1 \otimes H_2,\Psi}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ as follows. $\Psi(\alpha \otimes \beta) = \Psi_1(\alpha) \otimes \Psi(\beta)$. Given two consequence relations $\vdash_x$ and $\vdash_y$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Then define $\vdash_x \otimes \vdash_y := \vdash_{x \otimes y}$. If one the component logics is one-dimensional, we say the combined logic is a cell.*

The following lemma expresses an elementary property of the Hilbert space tensor product.

Lemma 11.1 *Suppose $\vdash_{x \otimes y} \alpha \otimes \beta$. Then $\vdash_x \alpha$ and $\vdash_y \beta$. Moreover, we have $\vdash_x \otimes \vdash_0 = \vdash_0$, i.e. combining any consequence relation with the inconsistent consequence relation yields the inconsistent consequence relation.*

Definition 11.3 *Given a Hilbert space logic $\mathcal{L}$ and a family of formulas $A = (\alpha_i)_{i \in I}$. Assume all $\Psi(\alpha_i)$'s are either mutually orthogonal pointers, i.e. the $\Psi(\alpha_i)$'s are mutually orthogonal rays, or the zero space. We assume at least one $\Psi(\alpha_i)$ to be non-zero. Then we say A is an observable. We think, in abuse of notation, of α_i as having the form $A = \lambda_i$, the λ_i's being the 'values' of the observable A.*

Obviously, the above definition is motivated by the physical concept of an observable represented by an Hermitian operator with non-degenerate eigenvalues. The above definition does not capture the analogue of the fact that the eigenvectors of such an operator form a complete orthonormal system. This is not necessary for our purposes. But it could be incorporated in a straightforward way. Eigenvectors are by definition non-zero. Note, however, that in the

above definition we allow for the inconsistent consequence relation, i.e. the zero vector, for reasons which will become obvious later.

We now define correlation of observables.

Definition 11.4 *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be Hilbert space logics. Let $A = (\alpha_i)_{i\in I}$ and $B = (\beta_i)_{i\in I}$ be observables of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Given a non-zero consequence relation $\vdash_{x\otimes y}$ of $\mathcal{L}_1 \otimes \mathcal{L}_2$. We say that $\vdash_{x\otimes y}$ is a pure correlation with respect to A and B if $\vdash_{x\otimes y} A = \lambda_i \otimes B = \lambda_i$ for some $i \in I$. We say that $\vdash_y$ is a correlation with respect to A and B if it is either a pure correlation with respect to A and B or a superposition of pure correlations with respect to A and B. We call a correlation of the latter kind an entanglement.*

Note that distinct pure correlations are orthogonal, i.e. the above definition of correlation makes sense.

11.1.5 Passing to the limit

We said that in our treatment of the measurement problem we need to reflect the 'classical' nature of the measuring instrument. As shown in Chapter 6 (Limiting Caseec Theorem), classical logic appears as a limiting case in the framework of holistic logics. Let us therefore, at this point, recall this process of 'passing to the limit'.

Passing to the limit logically

By the Limiting Case Theorem the limiting case of a holistic logic may be viewed as a complete classical theory or a family of complete classical theory. The 'direction' of passing to the limit was from non-monotonicity to monotonicity. To recall, The Limiting Case Theorem says that a holistic logic is monotonic if and only if it is of the form $\mathcal{L}_(\Sigma$ for some complete classical theory Σ.

Passing to the limit algebraically

Another direction of passing to the limit is from non-commutation of operators to their commutation. Again, we have that a holistic logic is commutative in the sense that all its revision operators commute must be of the the form $\mathcal{L}_\Sigma$ where Σ is a complete classical theory. So, algebraically, the direction of passing to the limit is from non-commutativity to commutativity.

Formulate this as a Proposition and give the simple proof. But it's preferable to state and prove it earlier in the book so that we can refer to it here..

Passing from Hilbert space to phase space

These ways of passing to a limit structure at the logical and algebraic level have a parallel at the level of physics. In classical mechanics, phase space plays the role of state space and is thus the analogue of Hilbert space in quantum mechanics. What is phase space? Let us for the sake of simplicity consider a single particle.

Then phase space is the set of pairs $\langle x(t), p(t)\rangle$, where $x(t)$ denotes position of the particle at time t and $p(t)$ its momentum at time t. A point in phase space contains all information about the state of the particle, since it allows us to compute any relevant classical physical quantity of the particle, for instance kinetic energy, angular momentum... Thus a point in phase space encodes all classical physical properties of the system. Logically speaking, it represents the (complete classical) theory Σ of the system or, put differently in our terminology of holistic logics, it represents the (holistic) logic $\mathcal{L}_\Sigma$. Thus on looking at phase space this way the transition from Hilbert space to phase space is smooth and natural. Logically, phase space, i.e. the state space in classical mechanics, may be viewed as the limiting case of a Hilbert space, which represents state space in quantum mechanics. We would like to remind the reader at this point of the way how the relation between phase space and Hilbert space is described in the Birkhoff-von Neumann paper (see Chapter 5.) In that paper this relationship is viewed in a completely different way , namely as a sort of analogy. In quantum mechanics, Birkhoff and von Neumann say, the state of a physical system is fully encoded in its wave function in analogy to the way that the state of a classical system of particles is known if position and momentum of each particle is known. The state space of the quantum system should thus, according to Birkhoff-von Neumann, be reprented by a function space, namely the Hilbert space of the wave functions of the system.

11.1.6 Complete classical theories and one-dimensional Hilbert space logics

We now make the simple connection between holistic logics of the form $\mathcal{L}_\Sigma$, i.e. complete classical theories, with Hilbert space logics. We can represent a complete classical theory as a one-dimensional Hilbert space logic in a natural way.

So given a logic $\mathcal{L}_\Sigma$ for some complete classical theory Σ. We have $\mathcal{L}_\sigma = \langle \mathcal{C}, F_\Sigma, \rightarrow\rangle$, where $\mathcal{C}_\Sigma = \{\vdash_\Sigma, 0\}$, $\vdash_\Sigma$ is defined by: $\alpha \vdash_\Sigma \beta$ if $\alpha \rightarrow \beta \in \Sigma$. Now, $\mathcal{L}_\Sigma$ can be presented as a one-dimensional Hilbert space logic as follows. Given a one dimensinal Hilbert space. Then the lattice of closed subspaces consists of two elements only, $Sub(H) = \{< x >, \{0\}\}$. Define the function $\Psi : FmlSub(H)$ by: $\Psi(\alpha) =< x >$ if $\alpha \in \Sigma$, $\Psi(\alpha) = \{0\}$ else. Then $\mathcal{L}_\Sigma = \mathcal{L}_{H,\Psi}$. We will, from now on, view $\mathcal{L}_\Sigma$ as the one-dimensional Hilbert space logic defined above. i.e. we omit the subscript Ψ.

As already mentioned, we call a tensor product of a one- dimensional Hilbert space logic with an arbitrary Hilbert space logic a *cell*

We may think of Σ as the 'theory' of a classical physical system, i.e. the set of statements or formulas true of this system. Such a formula may have the form $A = \lambda$ with the intuitive meaning that a the observable A pertaining to a classical system *has* the value λ. In this case we have $A = \lambda \in \Sigma$ or, equivalently, this says that $A = \lambda$ is a true statement about this system. Note that for any $\mu \neq \lambda$ $A = \mu$ is not in Σ or, equivalently, is a false statement about the system.

Let us reformulate this as

Lemma 11.2 *Given $\mathcal{L}_{\Sigma,\Psi}$. Then the following conditions are equivalent:*

- *(i) α is 'true'*
- *(ii) $\alpha \in \Sigma$*
- *(iii) $\vdash_x \alpha$*

It makes thus sense to say that a formula is is 'true' or 'false' in a one-dimensional Hilbert space logic according to our intuition that it represents a classical system and any statement about that system is either true or false.

It is easy to see that the following holds. It reflects our intuition that an observable of a classical system *has* exactly one value.

Lemma 11.3 *Let $\mathcal{L}$ be a one-dimensional Hilbert space logic and $(A = \lambda_i)_{i \in I}$ an observable of $\mathcal{L}$. Then there exists a (unique) $j \in I$ such that $A = \lambda_j$ is true in $\mathcal{L}$ and $A = \lambda_i$ is false in $\mathcal{L}$ for $i \neq j$.*

The limiting case results allow us to look at the logics of the form $\mathcal{L}_\Sigma$ as limiting cases in a twofold way. We may say that, given a holistic logic, then the limit of this logic is of the form $\mathcal{L}_\Sigma$, but we may also regard the limits of any of its consequence relations as being of this form. For Hilbert space logics this means that we may either say that the limit of a Hilbert space logic is either a single one-dimensional Hilbert space logic or a family of one-dimnsional Hilbert space logics. From the physical point of view the latter version seems preferable since it reflects the fact that phase space is the classical analogue of Hilbert space and phase space may, as explained above, be viewed as a family of complete classical theories.

Let us come back to the crucial question of the 'classical' nature of the measuring instrument. How are we to reflect this in our logical framework? We do not have much freedom in this. But there exists a natural way of doing it. In the light of the above there are two possibilities. Either we treat te measurement instrument as a single one-dimensional Hilbert space logic or a family of one-dimensional Hilbert space logics. We opt for the latter version for reasons which will become obvious later. We call such a family, reminiscent of 'phase space' in classical mechanics, a phase logic. Thus phase logics are families of the form $(\mathcal{L}_i)_{i \in I}$ each member being a one-dimensional Hilbert space logic.

We now have to define the concept of an observable for phase logics. As before it has to be a family of formulas of a certain kind.

Definition 11.5 *Let $\mathcal{L} = \mathcal{L}_{\Psi_i)_{i \in I}}$ a phase logic. Let $(A = \lambda_i)_{i \in I}$ be a family of formulas. We say that A is an observable for the phase logic if it is an observable of all its members.*

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We have now defined the notion of an observable both for Hilbert space logics and the limiting case of phase logics.

What does this say intuitively? Think of the phase logic as the phase space of a certain particle and of A as the observable of position of this particle. Then the family $(A = \lambda_t)_{t \in T}$ is the trajectory of the particle, λ_t being the position at time t.

We now have to say how to combine Hilbert space logics with phase logics and how to correlate observables if one of the systems is a phase logic.

Definition 11.6 *Let $\mathcal{L}_{H,\Psi}$ be a Hilbert space logic and $\mathcal{L} = \mathcal{L}_i)_{i \in I}$ a phase logic. We define the combined system to be the family $(\mathcal{L} \otimes \mathcal{L}_i)_{i \in I}$. Call a family of this sort, i.e. a family of cells, a register. Given two observables $A = (\lambda)_{i \in I}$ and $B = (\lambda_i)_{i \in I}$ pertaining to these systems respectively. We say that a family of states $(\vdash_i)_{i \in I}$, $\vdash_i$ being in cell i, is a correlation of A and B if each member is a correlation of A and B viewed as observables in each cell.*

We now define measurement.

Definition 11.7 *Given a phase logic $\mathcal{L}_1 = (\mathcal{L}_i)_{i \in I}$ and $\mathcal{L}_2$ any Hilbert space logic. Let A and B be observables pertaining to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. A process of interaction between $\mathcal{L}_1$ and $\mathcal{L}_2$ is called a measurement of B by $\mathcal{L}_2$ with pointer observable A if the state of the combined system 'after interaction' is a correlation of A and B. The phase logic $\mathcal{L}_1$ is called the measurement instrument, B its pointer observable.*

The following lemma expresses an observation which is crucial to our approach to the measurement problem.. Recall that the problem with Schroedinger's cat was that that after 'measurement' the system consisting of the cat and the electron was in an entanglement which seems to be at odds with the facts. The next lemma says that cells do not contain entanglements.

Lemma 11.4 *Given a cell (of a certain register), i.e. $Z_i = \mathcal{L}_i \otimes \mathcal{L}$ with $\mathcal{L}_\rangle$ being one-dimensional and two observables A and B pertaining to $\mathcal{L}_i$ and $\mathcal{L}$ respectively. Then every correlation of A and B in Z_i is pure. In fact, there exists exactly one correlation and this correlation is pure. For the process of measurement this means that it ends up in a pure correlation, i.e. not in an entanglement.*

Proof. Entanglements are by definition genuine superpositions of pure correlations. What do pure correlations in a cell look like? Generally, a pure correlation is of the form $A = \lambda_i \otimes B = \lambda_i$ for some $i \in I$ (viewed as a consequence relation. But now recall that in a one-dimensional Hilbert space logic $A = \lambda_i$ is true for exactly one $i \in I$. That is $A = \lambda_i \otimes B = \lambda_i$ is non-zero for exactly one $i \in I$ and zero for the others. This means that a cell contains exactly one correlation, which then must be pure. ∎

11.1.7 Temporal evolution in measurement as correlating a Hilbert space logic with a phase logic

We have to say something concerning the logical analogue of the quantum mechanical 'law of temporal evolution' of a system. In quantum mechanics the temporal evolution of a system is described by a unitary transformation of its Hilbert space. Unitary transformations preserve, in particular, superpositions. This leads us, in our logical treatment, to the postulate that, a Hilbert space logic evolves over time in a way that preserves superpositions.

Our second postulate is to capture an essential feature of the process of measurement, namely the fact that after measurement the observable to be measured and the pointer observable are correlated. We already used the terms 'the state before measurement' and 'state after measurement'. Generally, we assume that at any time a system is in a certain 'actual state'. So the above terms mean 'actual state before measurement' and 'actual state after measurement'.

By definition the state after measurement is a correlation. Note that we defined correlation for observables A and B pertaining to arbitrary Hilbert space logics $\mathcal{L}_1$ and $\mathcal{L}_2$. In the special case of measurement one of these logics is one-dimensional, and in this case Lemma...? tells us that the correlation in which the system ends up after measurement is pure, i.e. no entanglement.

So, the (actual) state after measurement is a correlation. But, clearly, we have to say something about how the state after measurement is related to the state before measurement. It is plausible that the state after measurement is not any correlation but must depend on the state of the system before measurement in a certain way. Therefore we postulate the following which conforms with quantum mechanics.

Let $\vdash_y$ be the state of $_\in$ before measurement. Then the state after measurement is of the form $\vdash_x \otimes \vdash_y$, where $\vdash_x$ is a non-zero state of $\mathcal{L}_2$.

......The following is probably not necessary Given any state before measurement of the form $x\otimes \mid B = \lambda\rangle$ (Hilbert space notation). Then this state evolves into the correlation $\mid A = \lambda\rangle\otimes \mid B = \lambda\rangle$. If the state before measurement is a superposition states as above then the state after measurement is a superposition of the corresponding correlated states after measurement. So, state $\sum_i x\otimes \mid B = \lambda_i\rangle$ evolves into the state $\sum_i \mid A = \lambda_i\rangle\otimes \mid B = \lambda_i\rangle$. These postulates probably admit a more elegant formulation our logical framework than the one we gave above in this draft. We will work on this......

11.1.8 Disentanglement and projection in measurement

We now have to take a closer look at correlations, if we combine a phase logic with an arbitrary Hilbert space logic. This is what in our view happens in measurement. The following is a simple but crucial observation which we formulate as a theorem. It is an immediate consequence of the above lemma.

Theorem 11.1 *Let $\mathcal{L}$ be an arbitrary Hilbert space logic and $(\mathcal{L}_i\otimes\mathcal{L})_{i\in I}$ a phase logic. Recall that the combined system is a register, namely the family of cells*

$\mathcal{L} \otimes \mathcal{L}_{i\in I}$. Let $A = (\lambda_i)_{i\in I}$ and $B = (\lambda_i)_{i\in I}$ be observables pertaining to the two systems respectively, i.e. B is the pointer observable. Then for every cell all correlations between A and B are pure, namely of the form $A = \lambda_i \otimes B = \lambda_i$. In particular, no cell contains entanglements.

The above says that the combination of any Hilbert space logic with a one-dimensional one does not admit entanglements. So, intuitively, we may view the process of combining a (genuine) Hilbert space logic with a phase logic as a sort of 'disentanglement'. It seems that, at the physical level, this is exactly what happens in the process of measurement. We will see that this may also be viewed as projection or 'collapse of the wave function'.

Let us take a closer look at this.

Given two correlated observables A and B in a register $(\mathcal{L}_{H_i,\Psi_i} \otimes \mathcal{L}_{H,\Psi})_{i\in I}$. Then, after measurement, cell i is in a state $x \otimes y$, $x \in H_i$, $y \in H$ such that $\vdash_{x\otimes y} A = \lambda_i \otimes B = \lambda_i$. Note that this holds no matter in which state $\mathcal{L}_H$ was before measurement. By the postulates above after measurement the state of the (composite) system is a correlation and, in cells, the correlations are of the above form, namely pure.

What has happened? Assume that before measurement the state of the system to be measured was in a superposition $y = \sum_i c_i B =\mid \lambda_i\rangle$ (in abuse of notation.) In the general case of correlating two arbitrary Hilbert space logics our postulates for correlating observables imply in accordance with those of quantum mechanics that the state after correlation (measurement) is $\sum_i c_i \mid A = \lambda_i\rangle \otimes \mid B = \lambda_i\rangle$, i.e. a superposition of pure correlations. This is an entanglement. Now, as we saw above, in the special the case of a cell, i.e. when one of the Hilbert space logics is one-dimensional, a process of disentanglement takes place. Namely, the components of the entanglement $A = \lambda_i \otimes B = \lambda_i$ are no longer 'entangled' but are 'stored' so to speak in the cells of the register. So, the entanglement was 'disentangled' and its components were ' stored' in the register. In this sense what we called a register is in fact a storage device. It seems worth investigating what this view could mean for quantum computation.

In the following considerations we use, for the sake of convenience, the usual Hilbert space notation of quantum mechanics. It is clear It is clear how this is to be translated into the language of Hilbert space logics.

So, again, assume the system to be measured before measurement to be in the superposition

$$y = \sum_i c_i \mid B = \lambda_i\rangle$$

Say, before measurement, the pointer of the measuring instrument points to some value λ, i.e. $A_=\lambda$. Since the measuring instrument is 'classical', the pointer variable *has* some value λ_j for some $j \in I$,i.e. $A = \lambda_j$ is true. So the composite system is in state

$$\sum_i c_i \mid A = \lambda_j\rangle \otimes \mid B = \lambda_i\rangle$$

After measurement it is in the correlated state

$$z = \sum_i c_i \mid A = \lambda_i\rangle \otimes \mid B = \lambda_i$$

This is still in Hilbert space language. Let us now look at this in a precise way from the point of view of *Hilbert space logics.* So view the above as the *consequence relation* $\vdash_z$. The above equation says that $\vdash_z$ is a superposition of certain consequence relations in the sense of Definition ?, namely the consequence relations $\vdash_{x_i \otimes y_i}$ such that $vdash_{x_i \otimes y_i} A = \lambda_i \otimes B = \lambda_i$. We can now say what is special about this superposition (entanglement) in the case of measurement, i.e. in the case when one of the Hilberet space logics is one-dimensional. In this case all but one of the members of the family $(\vdash_{x_i \otimes y_i})_{i \in I}$ is non-zero. Namely by Lemma ? all but one of the $\vdash_{x_i}$'s are non-zero. This is the expression of the fact that classically an observable *has* exactly one value in a given state. Put differently, in classical mechanics the proposition $A = \lambda$ is, in a given state, true and for $\mu \neq \lambda$ the proposition $A = \mu$ is false. Therefore $\vdash_z$ is not a proper entanglement, rather it concides with one of the components. We have

$\vdash_z = \vdash_{x_j \otimes y_i}$ where j is the index for which $A = \lambda_j$ is true.

Switching back to Hilbert space notation we can say that as a result of correlation with a one-dimensional Hilbert space logic the superposition z above was 'projected' onto the component $\mid A = \lambda_j\rangle \otimes \mid B = \lambda_j\rangle$. So our way of looking at measurement provides an explanation of the projection postulate (collapse of the wave function).

Let us summarise. Recall we regard measurement as correlating two obdervables A and B one of which say A pertaining to a phase logic $(\mathcal{L})_\rangle)_{\rangle \in \mathcal{I}}$ and B pertaining to an arbitrary Hilbert space logic $\mathcal{L}]$. The combination of the two logics is a register, namely the family $(\mathcal{L}_i \otimes \mathcal{L})_i$. The members of this family are called cells. We proved that cells do not admit entanglements. In fact, after measurement cell i is in state $\vdash_{x_i \otimes y_i}$ with $\vdash_{x_i} A = \lambda_i$ and $\vdash_{y_i} B = \lambda_i$.

So, on this view, the result of measurement is 'the contents' of the register, i.e. a *family* of the form $(A = \lambda_i \otimes B = \lambda_i)_{i \in I}$ rather than a single member of such as a family. Though appealing from a formal point of view this is in contrast to our intuition of measurement as yielding a single value for the observable to be measured rather than the family of all values it can assume. We could in our model account for this by assuming that there is only one cell of the register which is *actualised* in measurement in the sense that its value 'appears on the screen'. This cell would contain the 'actual' state of the combined system as we always assumed in our previous considerations that a system possesses an actual state. This process of actualising a value would then be governed by the statistical formalism of quantum mechanics. But this is beyond the realm of our logical framework. All the logical framework can account for is how the process of *disentanglement*, *projection* and *storage* involved in measurement works.

There is, however a striking parallel between this logical model and Everett's many worlds interpretation of quantum mechanics. According to Everett's approach all values that an observable can assume *are* realised in measurement, however in different worlds.

To get a more intuitive picture imagine one gets the task of somehow 'implementing' the model say as a computer program. In order to do this one would

have to implement the register as a data structure so to speak. One would, moreover, have to implement the process of disentanglement and storage of the 'entangled' components in the register. Another part of this implementation would then be 'actualisation', i.e. the process of (randomly) selecting a certain cell of the register the contents of which constitutes the 'actual value'. On this view of the model as being 'implemented' in nature the (Hilbert space) logics involved, in particular the register and its cells, appear as physically real. Again, we are reminded here of a certain analogy with the many worlds interpretation of quantum mechanics according to which all the diverse worlds into which the actual world splits in the process of measurement are considered equally real.

11.1.9 Classical Measurement and the Idempotence of Measurement

Let us now look at our model of measurement in the case of two classical systems. It should reflect the 'unproblematic' nature of classical measurement. In the case of classical measurement we have two classical systems. Both the measuring system and the system to be measured are classical and thus represented as one-dimensional Hilbert space logics. We will first analyse this case and then make the point that repeated measurement is very mich like classical measurement leaving the system to be measured unchanged. This is *idempotence* of measurement.

So let $(\mathcal{L}_i)_{i\in I}$ be the phase logic representing the measurement instrument and $\mathcal{L}$) the logic to be measured. Let $A = (\lambda_i)_{i\in I}$ be the pointer variable and $B = (\lambda_i)_{i\in I}$ the observable to be measured. Now note that there exists exactly one $j \in I$ such that (in Hilbert space notation) $\mid A = \lambda_j\rangle\otimes \mid B =$j $\rangle$ is a correlation (in cell j. What is j? It is the unique index j such that $B = \lambda_j$ is true in $calL$. The correlation (in cell j is then $\mid A = \lambda_j\rangle\otimes \mid B = \lambda_j\rangle$. We view of the fact that the second $\otimes$ -factor is the state of the system measured as expressing the fact that there was 'no change of state' in the system measured.

Let us now look at the case of repeated measurement. We already saw that we have an explanation of the fact that on measurement there was 'collapse' (projection). What our model of measurement still needs to explain is that any further measurement does not change the state of the system measured. This is *idempotence* of measurement. This follows from our second postulate concerning the temporal evolution in measurement as follows. Assume the state of $\mathcal{L}$ after measurement (and thus projection) is $\mid B = \lambda_j$ for some $j \in I$. Now perform a further measurement, i,e. correlate with the same measuring instrument as in the preceding measurement. We claim that the combined system ends up in state $A = \lambda_j\rangle \otimes B = \lambda_j\rangle$. This is a correlation as required. Why is it the correlation the measurement ends up with? The answer is that all other correlations violate the second principle. Namely, all correlations have the form $\mid A = \lambda_i\rangle\otimes \mid B = \lambda_i\rangle$. But, clearly, if $\mid B = \lambda_j\rangle$ was the initial state of $\mathcal{L}$, then $\mid A = \lambda_j\rangle\otimes \mid B = \lambda_j\rangle$ is the only corelation satisfying the second postulate.

Our (logical) model of measurement, therefore, provides an explanation of the absence of collapse in classical measurement as well as the idempotence of

measurement in general.

11.1.10 Schroedinger's cat revisited

Let us now revisit Schroedinger's cat and look at it from the viewpoint of the theory presented above.

Let A be the observable representing 'the cats's life' assuming the values 1 and 2. $A = 1$ says "the cat is alive", $A = 2$ says "the cat is dead". Let B be the observable representing the direction of the (z-component of) the spin of the electron. $B = 1$ says "spin is up" and $B = 2$ says "spin is down".

Let the phase logic representing the cat be (in explicit notation) $(\mathcal{L})_i$, $i = 1, 2$. More precisely $\mathcal{L}_\rangle$ has the form $\mathcal{L}_{\Psi_i}$ $\Psi_i(A = i = \langle x \rangle$. This says that $A = 1$, i.e. "cat is alive", is true in $\mathcal{L}_1$ and thus $A = 2$, i.e. "cat dead", is false in $\mathcal{L}_1$. $A = 2$, i.e. "cat is dead", is true in $_2$ and $A = 1$, i.e. "cat is alive", is false in $\mathcal{L}_2$.

Let $\mathcal{L}$ denote the (two-dimensional) Hilbert space logic pertaining to the electron (spin).

According to our general theory there are only two correlations of A and B, again in Hilbert space notation, $\mid A = 1\rangle \otimes \mid B = 1\rangle$ or in the notation of ..? $\mid alive\rangle \otimes \mid up\rangle$, and $\mid A = 2\rangle \otimes \mid B = 2\rangle$, i.e. $\mid dead\rangle \otimes \mid\rangle$ 'stored' in the register $(\mathcal{L}_i \otimes \mathcal{L}))_i$, $i = 1, 2$. So, in our model, the only possible states after measurement are the above, as we expect on the basis of our (macroscopic) experience of measurement. On this view the cat cannot be 'half alive' and 'half dead' but must be either alive or dead nor is the electron in a superposition. Rather the electron spin is either 'up' or 'down'

We may view the mechanism involved as follows.

Before measurement the state is (in Hilbert space notation)

$$z = \mid alive\rangle \otimes 1/\sqrt(2)(\mid up\rangle - \mid down\rangle)$$

or equivalently

$$z_1 = 1/\sqrt(2)(\mid alive\rangle \otimes \mid up\rangle - \mid dead\rangle \otimes \mid down\rangle)$$

Viewed logically the above says that the state x in the logical sense, i.e. $\vdash_{z_1}$ is a superposition of x_1 and y_1 such that $\vdash_{x_1} \mid alive\rangle \otimes \mid up\rangle$ and $\vdash_{y_1} \mid alive\rangle \otimes \mid down\rangle$.

After measurement we have a correlation, namely

$$z_1 = 1/\sqrt(2)(\mid alive\rangle \otimes \mid up\rangle + \mid dead\rangle \otimes \mid down\rangle$$

Viewed logically this means that $\vdash_{z_1}$ is a superposition of $\vdash_{x_2}$ and $\vdash_{y_2}$ such that $\vdash_{x_2} \mid alive\rangle \otimes \mid up\rangle$ and $\vdash_{y_2} \mid dead\rangle \otimes \mid down\rangle$. Both $\vdash_{x_2}$ and $\vdash_{y_2}$ are in $calL_1 \otimes \mathcal{L}$ as well as in $_2 \otimes \mathcal{L}$. However, both in $calL_1 \otimes \mathcal{L}$ and in $\mathcal{L}_2 \otimes \mathcal{L}$ exactly one of them zero and exactly one of them non-zero. Thus, from the logical point of view, $vdash_{z_2}$ is in fact equal to $\vdash_{x_2}$ or $\vdash_{y_2}$ and thus no genuine superposition.

This is how Schroedinger's cat looks in our picture.

11.1.11 Is the Hilbert space formalism the whole story? Legget's macrorealism

In several papers Anthony Legget, distinguished physicist and Nobel laureate, put forward his view on physical reality and the formalism of quantum mechanics which has become known as 'macrorealism'. Here are some quotations from [37].

"There is simply no convincing evidence that macroscopic superpositions of the type. .. exist in nature"

"A second reason for reluctance to consider the possibility outlined above lies at a more philosophical level. With few exceptions (who include David Bohm), scientists of the last 300 years or so have been deeply committed to a form of reductionism which holds, in effect, that the behaviour of a complex system of matter must be simply the the sum of the behaviour of its constituent parts."

"In this essay I shall try to defend three claims. the first is that the classic quantum measurement paradox, so far from being a non-problem, is a sufficiently glaring indication of the inadequacy of quantum mechanics as a total world view that it should motivate us actively to explore the likely direction in which it will break down."

"Indeed, Bohr, and with greater sophistication Reichenbach were able to to develop an interpretation of the quantum-mechanical formalism which is consistent with within its self-imposed limits precisely by postulating a radically different ontological status for microscopic entities such as electrons or neutrons and the macroscopic apparatus which perfoms the measurement. In the words of a famous quatation by Bohr: 'Atomic systems should not even be thought of as possessing definite properties in the absence of a specific experimental set-up designed to measure these proprties."

"The point of view I am proposing - namely that QM may not be the whole truth about the physical world- is likely to be strongly antithetical to the views of many ..."

So, according to Leggett the (Hilbert space) formalism of quantum mechanics gives a correct description of microscopic reality, the world of subatomic particles, quantum reality so to speak. He doubts however, that it provides an adequate description of macrosopic reality such as cats and buildings or planetes. On this view, it is the laws of classical physics, more precisely the theory of relativity, that govern macroscopic reality.

The crucial question is whether there exist 'macroscopic superpositions'. Can Schroedinger's cat be "half alive and half dead"? Something like this has never been observed. The explanation given by those believing in the universal validity of the formalism of quantum mechanics is a phenomenon called decoherence. We need not go into this in detail here. It is enough to say here that decoherence is a quantum effect arising from the interaction of macroscopic reality with its environment to the effect that superpositions of macroscopic states cannot be observed. So, on this view, even in the macroscopic world superpositions do exist, but, due to decoherence, they cannot be observed.

Let us, in this section, describe some parallels between the approach taken in this book, in particular our treatment of measurement, and Legget's views.

These parallels are surprising because both approaches stem from completely different origins. Legget's macrorealism rests on purely physical considerations whereas the approach of this book has its roots in logic. In fact, purely logical results (on holistic logics) and our (logical) treatment of the measurement problem suggest, at the physical level, a view very much along the lines of Legget's macrorealism. In particular, our results suggest an answer to the question whether the Hilbert space formalism is the whole story. This answer is negative. Moreover, our results suggest an answer to the question what the whole story is.

Once one has adopted macrorealism one is tempted to give a ready made answer to that question by saying that the whole story is "quantun mechanics + classical mechanics" rather than quantum mechanics (Hilbert space formalism). What is unsatisfactory about this answer is that this framework, "Hilbert space + phase space" does not constitute a unified framework. Rather, this answer amounts to saying that there are two different types of reality, i.e. quantum and classical, which have to be described in two completely different (mathematical) frameworks. This is unsatisfactory. A satisfactory answer to the question what the whole story is must provide a unified framework treating these two types of reality in a unified way. We will argue that from the logical point of view there is such a unified framework.

What is the main link between the logical framework of this book and Legget's macrorealism?

To explain this let us recall the train of thought underlying our discussion of the measurement problem. In our treatment of measurement we faced the problem how to represent the 'classical' nature of the measuring instrument. In this choice we were guided by the Limiting Case Theorem. We represent measurement instrument, logically, by a complete classical theory, more precisely by a family of complete classical theories. We combined in our treatment of the measurement problem Hilbert space with phase space. And, on certain reasonable assumptions in accordance with those of Quantum Mechanics, this approach provided us with a satisfactory way of accounting for the measurement problem.

holistic logics. The latter are, as we proved, closely related to Hilbert spaces. It is via this logical connection that two seemingly unrelated structures, namely phase space, which is the state space in classical mechanics, and Hilbert space, which is the state space in quantum mechanics, appear closely related in a natural way. As already explained in section ?, from the logical point of view phase space is a limiting case of Hilbert space. Logically, phase space is the 'monotonic limit' of Hilbert space. Since this is important for the point we want to make in this section let us recall what this means more precisely. The logics presentable by Hilbert spaces are non-monotonic. Now, the property of non-monotonicity may be ragarded as the logical counterpart of the presence of uncertainty relations at the physical level. Monotonicity of logic,accordingly, is the logical counterpart of the absence of uncertainty relations at the physical level. If, logically, we 'pass to the limit' in the direction of monotonicity we get classical logical structures, namely complete classical theories. Physically,

we may regard this 'passage to the limit' as being from uncertainty relations to the absence of uncertainty relations and we may think of these classical theories as the 'logics' of classical physical systems, i.e. sets of propositions true of a classical physical system. Thus, in this sense, we may view phase space as a limiting case of Hilbert space if we pass to the limit from uncertainty relations to the absence of uncertainty relations. This is the situation we have in classical mechanics.

On this view the transition from general holistic logics, in particular its main representatives, Hilbert space logics, to complete classical theories appears smooth. Moreover, complete classical theories *are* holistic logics, namely the monotonic ones.

Now, we can ask the question what all this could mean for physics? Of course, in this we can only speculate. But let us in the light of our results make a guess.

There is overwhelming evidence that the formalism of quantum mechanics, the core of which is Hilbert space, is adequate for describing the microscopic world. It undoubtedly provides the adequate description of electrons and other elementary particles. It is the measurement problem, especially in its dramatic form as 'Schroedinger's cat', however, that raises the question whether this formalism offers an adequate description of macroscopic reality such as the planets or cats.

From our logical point of view we may say that Hilbert space logics 'represent' microscopic reality whatever this means. In any case it is the reality of electrons etc. Now, if there is such a close conection between Hilbert space logics and classical logic and such a smooth transition, could it be that the limit structures, which are classical, represent some domain of reality too? In view of the role of phase space as a limiting case, could it be that the issue at stake is not "either Hilbert space or phase space" but "Hilbert and phase space"? Could it be that the limiting case is realised too in nature? If we answer these questions in the affirmative we arrive at Legget's macrorealism.

As we said, any answer to the question is speculation. But let us for the sake of an argument in favour of a positive answer return to our discussion of measurement. Recall that in our attempt to treat measurement we faced the problem of representing the measuring instrument. We had to account for the 'classical' nature of the measuring instrument. The way we chose to do this was to represent the measuring instrument by a limiting case structure, namely a complete classical theory and, in fact, this was the only reasonable choice we had. But let us also note that this way of doing it gave us -on certain natural assumptions also made in quantum mechanics- a satisfactory treatment of the measurement problem. Essentially, what we did was to combine a classical (holistic) logic with a non-classical one, a monotonic one with a non-monotonic one with the monotonic (classical) one representing the classical part in the measurement process, namely the measuring instrument. Formally this way of dealing with the measurement problem was successful. in that it provided an explanation that the state after measurement is not an entanglement and it provided an explanation for the projection postulate. The question we now face

is whether is not a purely formal trick or whether could be that way in nature. Again, all we can say is that if nature is that way then measurement poses no problem. This would in fact mean that the Hilbert space formalism is not the whole story. The whole story would be our framework of holistic logics including the limiting case structures. In a more familiar terminology the it would not be "Hilbert space" but "Hilbert space + Phase space". And the whole of (physical) reality would be not "quantum" but "quantum + classical".

Of course Leggett's macrorealism raises problems. First, if he were right, we would have completely different, even seemingly contradictory descriptions of the microscopic and the macroscopic physical world respectively. But what is macroscopic? What is microscopic? Second, it would mean that, since macroscopic objects may be regarded as aggregates of microscopic objects, classical mechanics should be reducible to quantum mechanics which, however, doesn't seem to be the case.

The question is whether the state of a macroscopic object can be a superposition. Can Schroedinger's cat be "half alive and half dead"? Something like that has never been observed. The explanation given by those believing in the universal validity of the formalism of quantum mechanics is a phenomenon called decoherence. We need not go into this in detail here. It is enough to say this here. Decoherence is a quantum effect arising from the interaction of macroscopic reality with its environment to the effect that superpositions of macroscopic states cannot be observed. So, on this view, even in the macroscopic world superpositions do exist, but, due to decoherence, they cannot be observed.

Another problem raised by Legget's macrorealism is the general problem of reductionism. Can the behaviour of macroscopic objects in principle be reduced to that of microscopic objects? If there are no macroscopic superpositions, then the answer is no because,if if the formalism of quantum mechanics applies, we never get rid of superpositions. The cat's being 'half alive and half dead' cannot be ruled out as a possible state of the cat.

11.1.12 Does logic depend on decoherence?

The explanation offered for the apparent absence of superpositions in our macroscopic experience by those who adhere to the view that the formalism of QM applies universally is, as already mentioned, a quantum mechanical effect called decoherence. On the other hand it seems that absence of superpositions is a precondition for classical (predicate) logic in which it is taken for granted that objects *possess* definite properties. If the adherents to decoherence were right, would this then mean that the possibility of classical logic is due to a a quantum effect? This question is at least worth discussing.

11.2 A Bit of Metaphysics

11.2.1 Dualism versus Monism in Physics and Logic

One of the fundamental differences between classical and quantum physics undoubtedly concerns the role of observation. In classical physics observation or, say, measurement is a sort of 'looking' at a certain objective reality outside the observer. It is taken for granted that there exists a clear cut separation between the observer and the reality observed. It is undisputable, however, that this dualistic picture is hard to defend in quantum mechanics. In Bohr's words, the process of measurement is "unanalysable". In the process of measurement in quantum mechanics the observer and the observed system form an inseparable whole. The dualistic view of the observer on the one hand and the observed reality on the other, of subject on the one hand and the object on the other, seems to be untenable in quantum mechanics.

In modern logic we have a similar dualism which is not even confined constitutes a typical feature of modern style logic in general since Tarski. It concerns the separation between the two components a modern logical system generally consists of, namely the separation between syntactic representation on the hand and semantic representation of the other. This issue of separation between syntax and semantics is addressed by Girard in [] as follows: "The current explanation of logic distinguishes between the world (objective) and its representation (subjective), the *object* and the *subject*. Logical realism relies on an opposition between *semantics* (the world) and *syntax* (its representation): this opposition is highly problematic...". There efforts under way which aim at overcoming this dualism, for instance in what has become known as game theoretic semantics.

In this book we have come across logical systems in which the problem of dualism is solved or at least avoided in a radical way. These are the logical structures we called holistic logics. How do holistic logics get rid of the problem?

In order to make things precise we first have to ask ourselves what the counterpart of this question is on the side of logic. This is closely related to the issue of semantics in logic. Suppose we have a logical system presented, say, as a deductive system. We are confronted with the issue of "reality" when it comes to giving a semantics to this system. It is in choosing the semantic structures that we make our commitment to a certain type of reality. The term 'giving semantics' shows by the way that we tacitly take it for granted that in building a system in the style of modern logic we require the system to have two separate components, namely a syntactic component and a semantic component, one describing the laws of reasoning so to speak and the other describing the domain of reality the reasoning is about. This issue of separation between syntax and semantics is addressed by Girard in a completely different context: "The current explanation of logic distinguishes between the world (objective) and its representation (subjective), the *object* and the *subject*. Logical realism relies on an opposition between *semantics* (the world) and *syntax* (its representation): this opposition is highly problematic...". One question we may ask is this. Is it possible to depart from classical logic in such a way that the resulting system is

free from this dualist feature which is displayed not only by classical logic but which, according to Girard, is characteristic of our current explanation of logic.

In any case, it seems that the way we reflect our intuitions concerning reality in quantum mechanics at the level of logic must be based on considerations concerning the role which semantics is to play in the systems to be constructed. In fact, our way of departing from classical logic in building quantum logic consists in imposing a condition on the system to be constructed which amounts to overcoming the dualism between syntactic and semantic representation.

There is another requirement, however, if the analogy *classical (quantum) mechanics versus classical (quantum) logic* is to be complete. Since classical mechanics is a limiting case of quantum mechanics we must require the structures we are looking for to have classical logic as a limiting case. So let us formulate our task as follows: *Define the concept of holistic logic in such a way that it has classical logic as a limiting case!*

Assume we succeed in this enterprise and assume further that the structures we come up with are strongly connected to the formalism of quantum mechanics (Hilbert space formalism), what would be gained? Well, we could then with substantial justification say that we touched upon the 'true' logical structures underlying quantum mechanics.

11.2.2 Logical Monadology

In chapter 3 we described Bohm's rheomode experiment with language. In that experiment he proposed to construct a new mode of language in addition to the modes we have so far such as indicative, subjunctive, imperative,and, to a certain degree even developed a concrete view on this. Bohm's motivation for this was the issue of reality in quantum mechanics. This is our starting point too. Let us make the connection.

As already pointed out, we want our way of departing from classical logic towards quantum logic to reflect the way how quantum mechanics departs from classical mechanics. We said that this departure is of a profound nature in that it involves a revision of our traditional view of physical reality.

We saw that in building a modern-style logical system the issue of reality enters the stage via semantics. It is in the choice of the semantic structures for a logic that we make a commitment to the structure of reality. What will our commitment to reality be? The answer is that we want this commitment to be minimal. In what sense? In order to explain this, assume we have a logic L presented somehow, say as a deductive system. Giving semantics to the logic means specifying a class of semantic structure for L. Normally, these structures are 'external' to the logic 'L'. Think for instance of predicate logic presented as a deductive system, say in Hilbert, Gentzen style... The procedure of giving semantics to the system is well known: specify the well known relational structures (models) of predicate logic, define the notion of satisfaction (truth) of a formula in a model, prove soundness and completeness of the system ...

If we now claim that the logic thus obtained is not just a mathematical construction but constitutes a tool for actually reasoning about the world, then

we have made a commitment to our view of reality. Namely, we have committed ourselves to the view that the relational structures that we chose as our semantic structures properly represent or reflect reality, at least the type of reality we reason about in predicate logic.. We commit ourslves to the view that there exist individuals, that these individuals have properties represented as predicates, that there exist relations between undividuals and so on. Needless to say, this is the linguistic reconstruction of the fragmented world view which needs revision in the quantum domain. Let us now, at this stage, make clear how we want to depart from classical logic in building quantum logic. Let us reiterate that our motivation is the issue of reality in quantum mechanics. We cannot solve these problems nor did we try to do that. What we tried to convey is nothing but the wide spread feeling among physicists and philosophers of science that quantum mechanics departs from classical mechanics in a very profound way in that this process touches upon the very nature of reality. On the other hand we said that giving semantics to a logical system always involves a certain commitment to a certain view of reality. We are, however, unclear about the nature of 'quantum reality'. It seems to us that this is the *main dilemma we face in building quantum logic.*

Is there a way out of this dilemma?

We think there is and and for this we propose an experiment. We emphasise it's an *experiment*!

We want to construct logical structures with a minimum of commitment to the structure of (external) reality

Let us make precise what we mean by 'mimimum of commitment to the structure of reality'.

Since we can hardly deny the logic to be real, we think that the following Principle expresses a sort of minimality condition.

Principle of Monadicity: The only semantic structures relevant to the logics to be described are the logics themselves. If we think of a logical system as a system reasoning about some domain of reality, then for a systems satisfying the above principle, a logical monad so to speak, this reality is nothing but itself. Thus the only (domain of) reality such a logic can 'reason' about is itself. Our intuition is that these systems -yet to be constructed- reason about themselves only, about their own 'internal working' so to spek. It should be a sort of reasoning about their own reasoning.

This is indeed a radical way of getting rid of the problem of reality in logic. That the logic itself is real can hardly be denied, and thus this is a radical way of avoiding any too strong commitment regarding the nature of (quantum) reality.

We would like here, for methodological reasons, to emphasise the following. The fact that we aim at constructing logics not committed to any reality 'outside itself' does by no imply any claim on our part concerning the nature of physical reality. Rather, let us reiterate that what we are doing is performing an experiment on the platform of logic in which we do not have the vexing problem of a potentially too strong commitment to the structure of reality. We pursue this line and we will see what the outcome of the experiment is.

The above principle has several implications.

The first implication concerns the notion of truth. Generally, in the spirit of the correspondence theory of truth, the notion of truth of a formula in some logical language is defined as 'truth in a model', i.e. relative to the semantic structures for the logic. In view of the above principle our notion of truth will be a notion of *self-referential truth.*

We may further ask the question what formulas can be true or false in view of the above principle. Obviously these are the formulas that 'talk' about the logic, formulas making a statement about the logic, i.e. metastatements. This poses another problem if we are tmake sense of the monadicity principle. Namely, if metastatements are the only formulas that can be true or false, then the object language must be capable of expressing metastatements. Moreover, we must require it to be rich in expressive power with respect to metastatements, i.e. it must be able to express *all* statements that can be a made in the language which may reasonably be called *the* metalanguage of the logic. We thus expect the object language to contain the metalanguage of the logic.

And, if we require the logic, as usual, to be sound and complete, this means that we require it to be capable of proving all true metastatements and only those. And thus, if we require sundness, we require *self-referential soundness* and if we require completeness, we require *self-referential completeness.*

What do we expect to be the logic of the metastatements. Since the metastatement 'talk' about some reality, namely the logic itself, and we, in this, assume the correspondence theory of truth, we expect them to obey classical logic. This means that the logic of the metalanguage must be classical logic.

Let us first point out that our condition that classical logic should be a limiting case of holistic logic should, heuristically, be viewed as imposing a condition on the way we depart from classical logic in the process of constructing quantum logic. It should be viewed as saying that we should not deviate 'too far' from classical logic in that process.

We may, reminiscent of Leibniz's "Monadology", want to construct 'logical monads', logical systems which are so to speak as self-contained as Leibniz's monads. In fact, we will see that there exist striking parallels between the framework of holistic logics and Leibniz's "Monadology".

We start this section with a word of caution because it may easily be perceived as having a smell of metaphysics. And if there is one thing we do not want to do then this is to engage in metaphysical speculation. Rather we would like to direct the reader's attention to certain parallels with a famous philosophical treatise which may be regarded as being metaphysical speculation. This treatise is Leinniz's "Monadology". All we will do in this paragraph is to point out certain parallels between what Leibniz says about the monads and certain properties of holistic logics. We think that these parallels are surprising and, maybe, not accidental.

What are Leibnz's monads? In paragraph 1 he says: "Monads, which are our conncern here, are nothing other than simple substances" They have 'no parts'. Paragraph 3 reads: "Now where there are no parts, neither extension nor shape nor division is possible. These monads are the true atoms of nature. In a word, they are the elements of things". Monads are completely self-sufficient.

Paragraph 7 contains the famous sentence: "The monads have no windows through which anything can come in or go out".

Another property of the monads is stated in paragraph 18: "...They enjoy self-sufficiency (autarkia that renders them the source of their internal actions and makes them, so to speak, incorporeal automata"

Another famous property Leibniz attributes to the monads is that of being a mirror. Every monad 'mirrors' the whole universe. Paragraph 56 reads: "Now this interconnection and accommodation of every created thing to every other, of all to each, gives every simple substance relations that express all the others so that each one is a perpetual living mirror of the universe."

The reader may note the 'modern talk' which might easily be found in a popular scientific or seriously scientific or philosophical book on quantum mechanics emphasising the fatures of interconnectedness and wholeness of 'quantum reality'.

What, now, are the parallels with holistic logics? To see this let us recall the main features of the notion of a holistic logic. First, why did we call them holistic? The main reason for this was the phenomenon of encodedness which means that non-orthogonal consequence relations of a holistic logics. We restate this for the case of a Hilbert space logics. Let $\mathcal{L}_H$ be a (holistic) logic presented by a Hilbert space H,let $x, y \in H$ be non-orthogonal with pointers σ_x, σ_y respectively. . Then we have:

$$\vdash_x \alpha \text{ iff } \vdash_y \sigma_x \rightsquigarrow \alpha$$

and

$$\vdash_y \alpha \text{ iff } \vdash_x \sigma_x \alpha$$

In this sense $\vdash_x$ and $\vdash_y$ are encoded in each other. We may view this as a sort of 'mirroring' . Note that 'mirroring' in this sense does not in any way imply 'containing'. For $\vdash_x \alpha$ does not imply $\vdash_x$. What the above intuitively says is: If $\vdash_x$ proves α, then $\vdash_y provesnotnecessarily\alpha$ but the provability of α in $\vdash_y$. But in which sense does the metaphor 'monad' apply to holistic logics? In which sense may they be viewed as 'self-contained' (logical) atoms. It is the property of self-referential completeness that supports this metaphor. Let us recall what self-referential completenees means. Essentially, it means that the elements of a holistic logic are logical systems reflecting their metatheory at the object level in the sense that they can prove all true metastatements and only those metastatements which are true. This, however, is not the full story yet. Rather, we have the 'no windows' theorem saying that the set of all statements proved by a holistic logic is classically inconsistent and thus has no classical model. So, metaphorically, we may say that holistic logics are just about themselves without 'windows'. Here the picture of Leibniz's self-contained 'incorporeal automata' comes to mind.

Leibniz regards the monads not only as real but as the ultimate constituents of reality. Let us tentatively pursue the analogy between Leibniz's monads and our holistic logics a bit further. We saw that, in our logical picture, pure

quantum states, which in the Hilbert space formalism are represented as rays in a Hilbert space, appear as the logical monads described. What about ascribing physical reality to them? What would it mean? It would mean ascribing reality to the quantum states themselves. On the raditional view the quantum states are states *of* a certain physical system say the electron in the hydrogen atom. The primary physical reality , however, is on this view the electron and and the concept of a (quantum) state appears as derived from the concept of a particle. They are, in this picture, a sort of states of information about the particle which constitutes the primary physical reality. In this there is no conceptual difference with classical mechanics where the state of a particle is known if its position and momentum are known (phase space). The analogy with Leibnz's monads would suggest a different view, namely the view regarding the quantum states as physical reality prior to that of the particle. On this view it would be the 'logical monads' (quantum states) that constitute the electron. The concept of a particle would on this view be a derived concept, namely derived from that of a state (monad). But this is enough metaphysics.

11.3 Reflections on holicity

Let us revisite the intuitive considerations on what in a preliminary terminology we called *logical monads*. We argue that the concept of a holistic logic, the topic of this chapter, conforms to those intuitions.

We said that logical monads should in no way rely on any 'external reality'. The only semantic structure relevant to the logic was to be the logic itself. Do holistic logics meet this requirement? We think yes they do. Recall our notion of truth in a holistic logic. It is selfreferential truth. Only metastatements, i.e, statements talking about the logic itself, can be true or false. The requirement that the metalanguage may be regarded as a sublanguage of the object language has been made precise. We have a proof operator in the (object) language, namely $\sigma \rightsquigarrow ...$. Holistic logics are sound and complete in that they can prove all true formulas and only these. The true formulas are the true metastatements. This is what we called self-referential truth.

What is the logic of the metastatements? Since metastatements talk about some piece of reality, namely the logic itself, we expect them to 'obey' classical logic. In fact, formulas expressing metastatements behave classically in every respect.

What is the significance of the No Windows Theorem, which says that the set of formulas proved by a consequence relation of a holistic logic is classically inconsistent. What would it mean if these 'theories' were classically consistent? We would then, by the completeness theorem of classical logic, have a classical models, i.e. 'external' semantic structures. The consequence relations would reason about some 'external reality'. This would destroy the picture of the monads. We think that this fact, namely the classical inconsisteny of the consequence relations, together with self-referential soundness and completeness strongly support our intuitions. The Non Windows Theorems 'protect' the sys-

tems considered from any 'external reality'. One is reminded of what Leibniz in his "monadology" says about the monads: "The monads have no windows".

Note, however, the following. We saw that the metalanguage may be regarded as a sublanguage of the object language. What about the set of metastatements proved by the consequence relations? This set of formulas *is* classically consistent. It is the set of true statements about a certain 'reality', and that reality is the logic itself.

We will see in "Kochen-Specker-Schütte Revisited" that the sharpened no windows theorem imlies the existence of a Kochen-Specker-Schütte tautology. So, on this view the Kochen-Specker-Schütte phenomenon loses its seemingly accidental nature. It appears as an exression of an essential feature of the logics underlying quantum mechanics.

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