

# Component structure induced by a random walk on a random graph.

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## Abstract

We consider random walks on two classes of random graphs and explore the likely structure of the vacant set viz. the set of unvisited vertices. Let  $\Gamma(t)$  be the subgraph induced by the vacant set. We show that for random graphs  $G_{n,p}$  (above the connectivity threshold) and for random regular graphs  $G_r$ ,  $r \geq 3$  there is a phase transition in the sense of the well-known Erdős-Renyi phase transition. Thus for  $t \leq (1 - \epsilon)t^*$  we have a unique giant plus logarithmic size components and for  $t \geq (1 + \epsilon)t^*$  we only have logarithmic sized components.

## 1 Introduction

The problem we consider can be described as follows. We have a finite graph  $G = (V, E)$ , and a simple random walk  $\mathcal{W}_u$  on  $G$ , starting at  $u \in V$ . What is the likely component structure induced by the unvisited vertices of  $G$  at step  $t$  of the walk?

Initially all vertices  $V$  of  $G$  are unvisited or *vacant*. We regard unvisited vertices as colored *red*. When  $\mathcal{W}_u$  visits a vertex, the vertex is re-colored *blue*. Let  $W_u(t)$  denote the position of  $\mathcal{W}_u$  at step  $t$ . Let  $\mathcal{B}(t) = \{W_u(0), W_u(1), \dots, W_u(t)\}$  be the set of blue vertices at the end of step  $t$ , and  $\mathcal{R}_u(t) = V \setminus \mathcal{B}_u(t)$ . Let  $\Gamma(t)$  be the subgraph of  $G$  induced by  $\mathcal{R}(t)$ . Initially  $\Gamma(0)$  is connected, unless  $u$  is a cut-vertex. As the walk continues,  $\Gamma(t)$  will shrink to the empty

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graph once every vertex has been visited. We wish to determine, as far as possible, the likely evolution of the component structure as  $t$  increases. In this paper we will consider two models of random graph: The random graph  $G_{n,p}$  and the random  $r$ -regular graph  $G_r$ ,  $r \geq 3$ , both on vertex set  $V = [n]$ .

Re-call the typical evolution of the random graph  $G_{n,p}$  as  $p$  increases from 0 to 1. Initially it consists of isolated vertices. As we increase  $p$  or equivalently add random edges, we find that the maximum component size increases from logarithmic size for  $p = c/n$ ,  $c < 1$  and then after the phase transition for  $c > 1$  there is jump in maximum component size to linear in  $n$  and all other components have logarithmic size. Our aim in this paper is to show that **whp**  $\Gamma(t)$  undergoes a reversal of this. In both cases  $\mathcal{G}(0)$  is connected and it will start to break up. There will be a critical value  $t^*$  such that if  $t < t^*$  by a sufficient amount then  $\mathcal{G}(t)$  consists of a unique giant component plus a collection of components of size  $O(\log n)$ . Once we have passed through the critical value, i.e.  $t > t^*$  by a sufficient amount then all components are of size  $O(\log n)$ . The maximum component size will then shrink to zero. We now provide some more details. We say that a graph is *sub-critical* if its maximum component size is  $O(\log n)$  and super-critical if it has a *unique* component of size  $\Omega(n)$  and all other components are of size  $O(\log n)$ .

We first consider  $G_{n,p}$ . Here we assume that

$$p = \frac{c \log n}{n} \text{ where } (c - 1) \log n \rightarrow \infty \text{ with } n \text{ and } c = n^{o(1)}.$$

Let

$$t_\theta = n(\log \log n + (1 + \theta) \log c).$$

**Theorem 1.** *Let  $\epsilon > 0$  be a small constant. Then **whp** we have (i)  $\Gamma(t)$  is super-critical for  $t \leq t_{-\epsilon}$  and (ii)  $\Gamma(t)$  is sub-critical for  $t \geq t_\epsilon$ .*

The bound  $c = n^{o(1)}$  is not optimal. On the other hand there is not going to be a phase transition if  $p$  is constant.

Now let us consider  $G_r$  for  $r \geq 3$ , constant.

Let

$$t^* = \frac{r(r-1) \log(r-1)}{(r-2)^2} n. \tag{1}$$

**Theorem 2.** *Let  $\epsilon > 0$  be a small constant. Then **whp** we have (i)  $\Gamma(t)$  is super-critical for  $t \leq (1 - \epsilon)t^*$ , (ii)  $\Gamma(t)$  is sub-critical for  $t \geq (1 + \epsilon)t^*$  and (iii) at some time  $t \sim t^*$  the maximum degree in  $\Gamma(t)$  is  $n^{2/3+o(1)}$ .*

We can also say something about the degree sequence of  $\Gamma(t)$ . Let

$$\rho = \frac{r-1}{r-2} \text{ and } N_t = ne^{-\frac{t}{\rho n}}.$$

Let

$$\tau_k = n^{1-1/k} \text{ and } t_k = \frac{\rho r n \log n}{k(r-2) + 2}. \quad (2)$$

Let  $\omega / \log \log n \rightarrow \infty$  and  $\omega = o(\log n)$ .<sup>1</sup>

**Theorem 3. Whp**

(a)  $|\mathcal{R}(t)| = (1 + o(1))N_t$ .

(b) 
$$D_s(t) = \begin{cases} (1 + o(1))N_t \binom{r}{s} p_t^s (1 - p_t)^{r-s} & \tau_{r-s} \ll t \leq t_{s+1} - \omega n \\ 0 & t \ll \tau_{r-s} \text{ or } t \geq t_{s+1} + \omega n \end{cases}$$

$$0 \leq s \leq r.$$

where  $D_s(t)$  is the number of vertices of degree  $s$  in  $\Gamma(t)$  and

$$p_t = e^{-\frac{(r-2)t}{\rho r n}}.$$

The range  $t_s - \omega n \leq t \leq t_s + \omega n$  contains the times when the number of vertices of degree  $s$  is constant in expectation and unlikely to be concentrated around its mean. We do not think that the asymptotic distribution is Poisson at this point. We can give some more information about the number of small components in  $\Gamma(t)$ . Again there is a gap containing the times when the expected number of such components is constant.

**Theorem 4.** *Let  $\epsilon$  be a small positive constant and let  $1 \leq k \leq \epsilon \log n$  and  $t \leq (1 - \epsilon)t_k$ . Let  $N(k, t)$  denote the number of components of  $\Gamma(t)$  with  $k$  vertices. Then **whp***

$$N(k, t) = \begin{cases} (1 + o(1)) \frac{r N_t}{k((r-2)k+2)} \binom{(r-1)k}{k-1} p_t^{k-1} (1 - p_t)^{k(r-2)+2} & \tau_{k(r-2)+2} \ll t \leq (1 - \epsilon)t_k \\ 0 & t \ll \tau_{k(r-2)+2} \text{ or } t \geq (1 + \epsilon)t_k \end{cases}$$

Note that we do not claim that we can prove that the statements in the above hold for all  $t$  **simultaneously**, although we suspect they do. Our proofs only show this to be true for most values of  $t$ .

## 1.1 Previous work

The only previous works on this subject that we are aware of are Benjamini and Sznitman [1], Windisch [18] and Černý, Teixeira and Windisch [5]. Papers [1], [18] deal with the component

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<sup>1</sup>In what follows,  $1 + o(1)$  means  $1 + O(\log^{-K} n)$  for all constant  $K > 0$ .

structure of the vacant set for a random walk on a  $d$ -dimensional torus. The latter deals with random walks on  $G_r$ . They show that **whp**  $\Gamma(t)$  is sub-critical for  $t \geq (1 + \epsilon)t^*$  and that there is a unique linear size component for  $t \leq (1 - \epsilon)t^*$ . They conjecture that  $\Gamma(t)$  is super-critical for  $t \leq (1 - \epsilon)t^*$  and we prove this conjecture. On the other hand their proof holds for *all* graphs in some class of regular graphs containing almost all regular graphs.

## 2 Uniformity

The main idea is to realise that the graph  $\Gamma(t)$  has a simple distribution. First consider  $G_{n,p}$ .

### Lemma 5.

*Consider a random walk on  $G_{n,p}$ . Conditional on  $N = |\mathcal{R}(t)|$ ,  $\Gamma(t)$  is distributed as  $G_{N,p}$ .*

**Proof** This follows easily from the principle of deferred decisions. We do not have to expose the existence or absence of edges between vertices in  $\mathcal{R}(t)$  until one of them is exposed.  $\square$

Thus to prove Theorem 1 we only need high probability estimates of  $|\mathcal{R}(t)|$ .

Now consider  $G_r$ . We give two structural definitions.

### Lemma 6.

*Consider a random walk on  $G_r$ . Conditional on  $N = |\mathcal{R}(t)|$  and its degree sequence  $\mathbf{d} = d_{\Gamma(t)}(v), v \in \mathcal{R}(t)$ ,  $\Gamma(t)$  is distributed as  $G_{N,\mathbf{d}}$ , the random graph with vertex set  $[n]$  and degree sequence  $\mathbf{d}$ .*

**Proof** Suppose that we condition on  $\mathcal{R}(t)$  and the *history*  $\mathcal{H} = (W_u(0), W_u(1), \dots, W_u(t))$ . If  $G_1, G_2$  are graphs with vertex set  $\mathcal{R}(t)$  and if they have the same degree sequence then substituting  $G_2$  for  $G_1$  will not conflict with  $\mathcal{H}$  i.e. every extension of  $G_1$  is an extension of  $G_2$  and vice-versa.  $\square$

Thus to prove Theorem 2 we only need high probability estimates of the degree sequence of  $\Gamma(t)$ . The proof of Theorem 4 can in principle be derived from this, although we do not have a simple way of doing it. Instead we rely on a further characterisation of  $\Gamma(t)$ .

We use the configuration or pairing model of Bollobás [3] and Bender and Canfield [4]. We start with  $n$  disjoint sets of points  $W_1, W_2, \dots, W_n$  each of size  $r$ . We let  $W = \bigcup_{i=1}^n W_i$ . A *configuration*  $F$  is a partition of  $W$  into  $rn/2$  pairs i.e. a *pairing*.  $\Omega$  is the set of configurations. If  $F \in \Omega$  defines an  $r$ -regular multi-graph  $G_F = ([n], E_F)$  where  $E_F = \{(i, j) : \exists \{x, y\} \in F : x \in W_i, y \in W_j\}$  i.e. we contract  $W_i$  to a vertex  $i$  for  $i \in [n]$ .

It is known that (i) each (simple) graph arises the same number of times as  $G_F$  and (ii) the probability that  $G_F$  is simple is bounded below by a constant, if  $r$  is constant. Thus if  $F$  is chosen uniformly at random from  $\Omega$  then any event that occurs **whp** for  $G_F$  will occur **whp** for  $G_r$ .

Suppose now that we generate a random  $F$  as we do a random walk on  $[n]$ . We begin with a starting value  $i_1$  and at the start of  $t$ th step we are at some  $i_t$  and we have a partition  $R_t, B_t$  of  $W$  into Red and Blue points respectively. Initially,  $R_1 = W$  and  $B_1 = \emptyset$ . In addition we have a collection  $F_t$  of disjoint pairs from  $W$ . Initially  $F_1 = \emptyset$ .

We choose  $x$  randomly from  $W_{i_t}$  – choose a random edge incident with  $i_t$ . If  $x \in R_t$  we choose  $y$  randomly from  $R_t \setminus \{x\}$  and add then let  $F_{t+1} = F_t \cup \{\{x, y\}\}$ . Suppose that  $y \in W_j$ . This is equivalent to moving from  $i_t \in \mathcal{B}(t)$  to  $i_{t+1} = j$ . We remove  $\{x, y\}$  from  $R$  and move them to  $B$ .

If on the other hand,  $x \in B$  then it has previously been paired with a  $y \in W_j \cap B$  and we move from  $i_t$  to  $i_{t+1} = j$  and let  $R_{t+1} = R_t, B_{t+1} = B_t$  and we let  $F_{t+1} = F_t$ .

After  $t$  steps we will have constructed a random collection  $F_t$  of  $\leq t$  disjoint pairs from  $W$  and in principle we can extend it to a random configuration  $F$  by adding a random pairing of  $R_t$  to it. We summarise what we need in a lemma.

**Lemma 7.**

- (a)  $F_t$  plus a random pairing of  $R_t$  is a random member of  $\Omega$ .
- (b)  $i \in \mathcal{R}(t)$  iff  $W_i \subseteq R_t$ .

□

### 3 Vacancy probabilities

As in our previous papers on random walks on random graphs, we make heavy use of Lemma 8 below. Let  $P$  be the transition matrix of the walk and let  $P_u^{(t)}(v) = \mathbf{Pr}(W_u(t) = v)$  be the  $t$ -step transition probability. We assume the random walk  $\mathcal{W}_u$  on  $G$  is ergodic, and thus the random walk has stationary distribution  $\pi$ , where  $\pi_v = d(v)/(2m)$ .

Let  $\lambda_2$  be the second eigenvalue of  $P$ , and let  $\Phi_G$  be the conductance of  $G$  i.e.

$$\Phi_G = \min_{S \subseteq V, \pi_S \leq 1/2} \frac{\sum_{x \in S} \pi_x P(x, \bar{S})}{\pi_S}.$$

Then,

$$1 - \Phi_G \leq \lambda_2 \leq 1 - \frac{\Phi_G^2}{2} \quad (3)$$

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_2^t. \quad (4)$$

A proof of this can be found for example in Jerrum and Sinclair [14]. In addition, Friedman [11] has shown that **whp**  $\lambda_2 \leq (2\sqrt{r-1} + \epsilon)/r \leq 29/30$ , say. (For (4) we need  $\lambda_2 = \lambda_{\max}$  which can be achieved by making the chain lazy i.e. by not moving with probability 1/2 at each step. This has no significant effect on the analysis).

Let  $T$  be such that, for  $t \geq T$

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| = O\left(\frac{\min_{x \in V} \pi_x}{n^3}\right). \quad (5)$$

In which case we can **whp** take

$$T \leq 120 \log n. \quad (6)$$

If inequality (5) holds, we say the distribution of the walk is in *near stationarity*. Fix two vertices  $u, v$ . Let  $h_t = \mathbf{Pr}(W_u(t) = v)$  be the probability that the walk  $\mathcal{W}_u$  visits  $v$  at step  $t$ . Let

$$H(z) = \sum_{t=T}^{\infty} h_t z^t \quad (7)$$

generate  $h_t$  for  $t \geq T$ .

We next consider the returns to vertex  $v$  made by a walk  $\mathcal{W}_v$ , starting at  $v$ . Let  $r_t = \mathbf{Pr}(W_v(t) = v)$  be the probability that the walk returns to  $v$  at step  $t = 0, 1, \dots$ . In particular note that  $r_0 = 1$ , as the walk starts on  $v$ . Let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate  $r_t$ , and let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \quad (8)$$

Thus, evaluating  $R_T(z)$  at  $z = 1$ , we have  $R_T(1) \geq r_0 = 1$ .

For  $t \geq T$  let  $f_t = f_t(u \rightarrow v)$  be the probability that the first visit made to  $v$  by the walk  $\mathcal{W}_u$  to  $v$  in the period  $[T, T+1, \dots]$  occurs at step  $t$ . Let

$$F(z) = \sum_{t=T}^{\infty} f_t z^t$$

generate  $f_t$ . Then we have

$$H(z) = F(z)R(z). \quad (9)$$

The following lemma gives the probability that a walk, starting from near stationarity makes a first visit to vertex  $v$  at a given step. For proofs of the lemma and its corollary, see [6], [8].

**Lemma 8.** *Let  $R_v = R_T(1)$ , where  $R_T(z)$  is from (8). For some sufficiently large constant  $K$ , let*

$$\lambda = \frac{1}{KT}, \quad (10)$$

where  $T$  satisfies (5). Suppose that

(i) For some constant  $\theta > 0$ , we have

$$\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \theta.$$

(ii)  $T\pi_v = o(1)$  and  $T\pi_v = \Omega(n^{-2})$ .

There exists

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \quad (11)$$

such that for all  $t \geq T$ ,

$$f_t(u \rightarrow v) = (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^{t+1}} + O(T\pi_v e^{-\lambda t/2}). \quad (12)$$

$$= (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^t} \quad \text{for } t \geq \log^3 n. \quad (13)$$

**Corollary 9.** *For  $t \geq T$  let  $\mathcal{A}_v(t)$  be the event that  $\mathcal{W}_u$  does not visit  $v$  at steps  $T, T+1, \dots, t$ . Then, under the assumptions of Lemma 8,*

$$\Pr_{\mathcal{W}}(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2\pi_v e^{-\lambda t/2}) \quad (14)$$

$$= \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} \quad \text{for } t \geq \log^3 n. \quad (15)$$

We use the notation  $\Pr_{\mathcal{W}}$  when we want to emphasise that we are dealing with the probability space of walks on  $G$ .

## 4 The evolution of $\Gamma(t)$ in $G_r$

### 4.1 Estimates of $R_v$

Let

$$\ell_1 = \epsilon_1 \log_r n$$

for some sufficiently small  $\epsilon_1$ . A cycle  $C$  is *small* if  $|C| \leq \ell_1$ . A vertex is *nice* if it is at distance at least  $\ell_1$  from any small cycle. Let  $\mathcal{N}$  denote the nice vertices and  $\bar{\mathcal{N}}$  denote the vertices that are not nice.

It is straightforward to prove by first moment calculations that:

$$\text{Whp there are at most } n^{2\epsilon_1} \text{ vertices that are not nice.} \quad (16)$$

$$\text{Whp there are no two small cycles within distance } \ell_1 \text{ of each other.} \quad (17)$$

A vertex  $v$  is *tree-like* to depth  $k$  if  $N_k(v)$  induces a tree, rooted at  $v$ . Here  $N_k(v)$  denotes the set of vertices at distance at most  $k$  from  $v$ ,  $k \geq 1$ . Thus a nice vertex is tree-like to depth  $\ell_1/2$ .

**Lemma 10.**

(a) *If  $v$  is nice then*

$$R_v = (1 + o(1))\rho \text{ where } \rho = \frac{r-1}{r-2}.$$

(b) *If  $v$  is not nice then*

$$R_v \leq (1 + o(1))\frac{r}{r-2}.$$

**Proof** (a) Let  $H$  denote the subgraph of  $G$  induced by  $N_{\ell_1/2}(v)$ . This is a tree and we can embed it into an infinite  $r$ -regular tree  $\mathcal{T}$  rooted at  $v$ . Let  $\mathcal{X}$  be the walk on  $\mathcal{T}$ , starting from  $v$ , and let  $X_t$  be the distance of  $\mathcal{X}$  from the root vertex at step  $t$ .

Let  $W_0 = 0$ , and let  $W_t$  be the distance from  $v$  of  $\mathcal{W}$  in  $G$  at step  $t$ . We note that we can couple  $\mathcal{W}_v, \mathcal{X}$  so that  $W_t = X_t$  up until the first time that  $W_t > \ell_1/2$ .

The values of  $X_t$  are as follows:  $X_0 = 0$ ,  $X_1 = 1$ , and if  $X_t = 0$  then  $X_{t+1} = 1$ . If  $X_t > 0$  then

$$X_t = \begin{cases} X_{t-1} - 1 & \text{with probability } q = \frac{1}{r} \\ X_{t-1} + 1 & \text{with probability } p = \frac{r-1}{r}. \end{cases} \quad (18)$$

We note the following result (see e.g. [10]), for a random walk on the line  $= \{0, \dots, a\}$  with absorbing states  $\{0, a\}$ , and transition probabilities  $q, p$  for moves left and right respectively. Starting at vertex  $z$ , the probability of absorption at the origin 0 is

$$\rho(z, a) = \frac{(q/p)^z - (q/p)^a}{1 - (q/p)^a} \leq \left(\frac{q}{p}\right)^z, \quad (19)$$

provided  $q \leq p$ .

Let  $U_\infty = \{\exists t \geq 1 : X_t = 0\}$ , i.e. the event that the particle ever returns to the root vertex in  $\mathcal{T}$ . It follows from (19) with  $z = 1$  and  $a = \infty$  that

$$\Pr(U_\infty) = \frac{1}{r-1}. \quad (20)$$

It follows that the expected number of visits by  $\mathcal{X}$  to  $v$  is  $\frac{1}{1-\frac{1}{r-1}} = \rho$ . We write

$$R_v = \sum_{t=0}^T r_t \text{ and } \rho = \sum_{t=0}^{\infty} \rho_t$$

where  $\rho_t = \Pr(X_t = v)$ .

Now  $r_t = \rho_t$  for  $t \leq \ell_1/2$  and part (a) follows from

$$\sum_{t=\ell_1/2+1}^T r_t = o(1) \text{ and } \sum_{t=\ell_1/2+1}^{\infty} \rho_t = o(1). \quad (21)$$

The first equation of (21) follows from

$$\left| r_t - \frac{1}{n} \right| \leq \lambda_{\max}^t$$

where  $\lambda_{\max}$  is the second largest eigenvalue of the walk. This follows from (4).

The second equation of (21) is proved in Lemma 7 of [6] where it is shown that

$$\sum_{t=\ell_1/2+1}^{\infty} \rho_t \leq \sum_{2j=\ell_1/2+1}^{\infty} \binom{2j}{j} \frac{(r-1)^j}{r^{2j}} \leq \sum_{2j=\ell_1/2+1}^{\infty} \left( \frac{4(r-1)}{r^2} \right)^j. \quad (22)$$

(b) We next note a property of random walks on undirected graphs which follows from results on electrical networks (see e.g. Doyle and Snell [9]). Let  $v$  be a given vertex in a graph  $G$  and  $S$  a set of vertices disjoint from  $v$ . Let  $p(G)$ , the *escape probability*, be the probability that, starting at  $v$ , the walk reaches  $S$  before returning to  $v$ . For an unbiased random walk,

$$p = \frac{1}{d(v)R_{EFF}},$$

where  $R_{EFF}$  is the effective resistance between  $v$  and  $S$  in  $G$ . We assume each edge of  $G$  has resistance 1. In the notation of this paradigm, deleting an edge corresponds to increasing the resistance of that edge to infinity. Thus by Raleigh's Monotonicity Law, if edges are deleted from  $G$  to form a sub-graph  $G'$  then  $R'_{EFF} \geq R_{EFF}$ . So, if we do not delete any edges incident with  $v$  then  $p' \leq p$ .

It follows from (17) that  $H$  becomes a tree after removing one edge. We can remove an edge not incident with  $v$ . By the above discussion on electrical resistance we see that this will not decrease  $\Pr(U_\infty^*)$ , where this is  $U_\infty$  defined with respect to  $\mathcal{T}^*$  which is  $\mathcal{T}$  less one edge, not incident with  $v$ . We can argue crudely that

$$\Pr(U_\infty^*) \leq \frac{1}{r} + \frac{r-1}{r} \cdot \frac{1}{r-1} = \frac{2}{r}.$$

This is because there is an  $\frac{r-1}{r}$  chance of a first move to a part of the tree that has branching factor  $r-1$  at every vertex.

Let  $U_1^* = \{\mathcal{X} \text{ returns to the root vertex after starting at } \ell_1/2\}$ . Then, with  $f_T$  equal to the probability of a return by  $\mathcal{W}_v$  to  $v$  during  $[1, T]$ , we have

$$f_T \leq \Pr(U_\infty^*) + T\Pr(U_1^*). \quad (23)$$

The RHS of (23) is at least the probability that  $\mathcal{W}_v$  returns before reaching distance  $\ell_1/2$  or returns after reaching distance  $\ell_1/2$  at some time  $t \leq T$ .

Now, using (19), we see that

$$\Pr(U_1) \leq \frac{1}{(r-1)^{\ell_1/4}}. \quad (24)$$

Here we have  $\ell_1/4$  in place of  $\ell_1/2$  to account for the one place where we move left with probability  $\frac{1}{r-2}$ . We argue that at least one of the paths from  $v$  to  $w$  or  $w$  to the boundary must be at least  $\ell_1/4$  and not use the vertex incident to the deleted edge.

Thus  $f_T \leq (2 + o(1))/r$  and since  $R_v \leq \frac{1}{1-f_T}$  we have  $R_v \leq \frac{r+o(1)}{r-2}$ . □

## 4.2 Proof of Theorem 3

We will initially assume that  $t \geq \log^3 n$  and deal with the very beginning of the walk in Section 4.5.

Corollary 9 gives the probability of not visiting a single vertex in time  $[T, t]$ . We need to extend this to certain small sets of vertices. In particular we only need to consider sets consisting of  $v$  and a subset of its neighbours  $N(v)$ . Let  $S$  be such a subset.

Suppose now that  $S$  is a subset of  $V$  with  $|S| = o(n)$ . By contracting  $S$  to single vertex  $\gamma = \gamma(S)$ , we form a graph  $H = H(S)$  in which the set  $S$  is replaced by  $\gamma$  and the edges that were contained in  $S$  are contracted to loops. The probability of no visit to  $S$  in  $G$  can be found (up to a multiplicative error of  $1 + O(1/n^3)$ ) from the probability of a first visit to  $\gamma$  in  $H$ . This is the content of Lemma 11 below.

We first check the mixing time of a walk on  $H$ . Note first that the conductance of  $H$  is at least that of  $G$ . This is because the set of values that we minimise over for  $H$  is a subset of the set of values that we minimise over for  $G$ . It follows that the mixing time for  $\mathcal{W}$  in  $H$  is also  $O(\log n)$ .

**Lemma 11.** [8] *Let  $\mathcal{W}_u$  be a random walk in  $G$  starting at  $u \notin S$ , and let  $\mathcal{X}_u$  be a random walk in  $H$  starting at  $u \neq \gamma$ . Let  $T$  be a mixing time satisfying (5) in both  $G$  and  $H$ . Then*

$$\Pr(\mathcal{A}_\gamma(t); H) = \Pr(\wedge_{v \in S} \mathcal{A}_v(t); G) \left(1 + O\left(\frac{1}{n^3}\right)\right),$$

where the probabilities are those derived from the walk in the given graph.

### Proof

Note that  $m = rn/2 = |E(G)| = |E(H)|$ . Let  $W_x(j)$  (resp.  $X_x(j)$ ) be the position of walk  $\mathcal{W}_x$  (resp.  $\mathcal{X}_x(j)$ ) at step  $j$ . Let  $\Gamma = G, H$  and let  $P_u^s(x; \Gamma)$  be the transition probability in  $\Gamma$ , for the walk to go from  $u$  to  $x$  in  $s$  steps.

$$\begin{aligned} \Pr(\mathcal{A}_\gamma(t); H) &= \sum_{x \neq \gamma} P_u^T(x; H) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; H) \\ &= \sum_{x \neq \gamma} \left(\frac{d(x)}{2m}(1 + O(n^{-3}))\right) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; H) \quad (25) \end{aligned}$$

$$\begin{aligned} &= \sum_{x \notin S} (P_u^T(x; G)(1 + O(n^{-3}))) \Pr(W_x(s-T) \notin S, T \leq s \leq t; G) \quad (26) \\ &= \Pr(\wedge_{v \in S} \mathcal{A}_v(t); G)(1 + O(1/n^3)). \end{aligned}$$

Equation (25) follows from (5). Equation (26) follows because there is a natural measure preserving map  $\phi$  between walks in  $G$  that start at  $x \notin S$  and avoid  $S$  and walks in  $H$  that start at  $x \neq \gamma$  and avoid  $\gamma$ .  $\square$

Fix  $v \in V$  and let  $N(v) = \{w_1, w_2, \dots, w_r\}$  and choose  $0 \leq s \leq r$ . We estimate

$$\begin{aligned} \Pi_s(v, t) &= \Pr_{\mathcal{W}}(v \text{ is a vertex of degree } s \text{ in } \Gamma(t)) \\ &= \binom{r}{s} \Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \dots, w_r\} \subseteq \mathcal{B}(t)) \quad (27) \end{aligned}$$

Thus

$$\mathbf{E}D_s(t) = \sum_{v \in V} \Pi_s(v, t).$$

The next lemma gives us enough information to compute the expected degree sequence of nice vertices in  $\mathcal{R}(t)$ .

**Lemma 12.**  $\text{Whp}_G$ , for all nice vertices  $v$ , (we use the subscript  $G$  when we want to emphasise that the probability space is random  $r$ -regular graphs).

(a)  $\Pr_{\mathcal{W}}(v \in \mathcal{R}(t)) = (1 + o(1))e^{-\frac{t}{\rho n}}$ .

(b)  $\Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \dots, w_r\} \subseteq \mathcal{B}(t)) = (1 + o(1))e^{-\frac{t}{\rho n}} p_t^s (1 - p_t)^{r-s}$ .

**Proof** Part (a) follows directly from Lemmas 8 and 10.

For part (b) we can write

$$\begin{aligned} & \Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \dots, w_r\} \subseteq \mathcal{B}(t)) \\ &= \sum_{X \subseteq [s+1, r]} (-1)^{|X|} \Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \cup X \subseteq \mathcal{R}(t)) \\ &= \sum_{X \subseteq [s+1, r]} (-1)^{|X|} \frac{1 + O(T\pi_{\gamma_X})}{(1 + p_{\gamma_X})^t}, \end{aligned} \tag{28}$$

where  $|X| = 0, \dots, r-s$ . If  $s = 0$ , we suppose that  $\{v, w_1, \dots, w_s\} = \{v\}$ . The term  $\gamma_X$  denotes the contraction of  $\{v, w_1, \dots, w_s\} \cup X$ , and

$$\pi_{\gamma_X} = O\left(\frac{1}{n}\right) \text{ and } p_{\gamma_X} = (1 + o(1)) \frac{((r-2)(s + |X|) + r)(r-2)}{r(r-1)n}.$$

**Explanation:** The expression for  $p_{\gamma_X}$  is the product of  $\frac{r(s+|X|+1)}{rn}$ , the total degree of  $\gamma_X$  divided by the total degree and the inverse of the expected number of returns to  $\gamma_X$  in the mixing time. Since  $v$  is nice, the expected number of returns (up to a factor  $1 + o(1)$ ) is  $\frac{1}{1-f}$  where  $f$  is the probability of return to the root in an infinite tree with branching factor  $r-1$  at each non-root vertex. At the root there are  $s + |X|$  loops and  $(r-2)(s + |X|) + r$  branching edges. This gives

$$f = \frac{2(s + |X|)}{(s + |X| + 1)r} + \left(1 - \frac{2(s + |X|)}{(s + |X| + 1)r}\right) \frac{1}{r-1} \text{ and } 1 - f = \frac{((r-2)(s + |X|) + r)(r-2)}{(s + |X| + 1)r(r-1)}.$$

Thus, for  $s = 0, 1, \dots, r$ ,

$$\begin{aligned} & \Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \dots, w_r\} \subseteq \mathcal{B}(t)) = \\ & \exp\left\{-\left(1 + o(1)\right) \frac{(r-2)^2 s + r(r-2)}{r(r-1)n} t\right\} \sum_{X \subseteq [s+1, r]} (-1)^{|X|} \exp\left\{-\left(1 + o(1)\right) \frac{(r-2)^2 |X|}{r(r-1)n} t\right\} \end{aligned} \tag{29}$$

When  $t = O(n)$  we can write (29) as

$$\exp\left\{-\left(1 + o(1)\right) \frac{(r-2)^2 s + r(r-2)}{r(r-1)n} t\right\} \left(1 - \exp\left\{-\frac{(r-2)^2}{r(r-1)n} t\right\}\right)^{r-s} + o(1).$$

and (b) follows, since the terms above are  $\Omega(1)$ . When  $t/n \rightarrow \infty$  we go back to (29) and observe that the sum is  $1 - o(1)$  and thus

$$\Pr_{\mathcal{W}}(\{v, w_1, \dots, w_s\} \subseteq \mathcal{R}(t) \text{ and } \{w_{s+1}, \dots, w_r\} \subseteq \mathcal{B}(t)) = (1 + o(1)) \exp \left\{ -\frac{(r-2)^2 s + r(r-2)}{r(r-1)n} t \right\}$$

as required.  $\square$

It follows from Lemmas 8 and 10 that

$$\mathbf{whp} \bar{\mathcal{N}} \subseteq \mathcal{B}(t) \text{ for } t \geq 10\epsilon_1 n \log n. \quad (30)$$

Lemma 12 verifies that  $N_t$  and  $D_s(t)$  have expectations asymptotic to the claimed values in Theorem 3. The expressions in this lemma only apply to nice vertices. But (30) allows us to use them for  $t \geq t_2$ . On the other hand, for  $t \leq t_2$  we have

$$\mathbf{E}(D_s(t)) \geq n^{1/2} \gg n^{2\epsilon_1}$$

and the vertices of  $\bar{\mathcal{N}}$  are asymptotically negligible. We will therefore ignore them in computations concerning  $D_s(t)$  from now on.

In particular Lemma 12 and the Markov inequality already show that

$$D_s(t) = 0 \text{ whp for } s > 0 \text{ and } t \geq t_{s+1} + \omega n \text{ or } t \ll \tau_{r-s}. \quad (31)$$

Thus **whp**  $\Gamma(t)$  consists of isolated vertices only, from  $t_1$  until the cover time.

We now need to prove concentration for  $N_t$  and  $D_s(t)$ .

Let  $\mathcal{R}_{\mathcal{N}}(t) = \mathcal{R}(t) \cap \mathcal{N}$  and  $Z_{\mathcal{N}}(t) = |\mathcal{R}_{\mathcal{N}}(t)|$ . We will use the Chebyshev inequality. Suppose that  $t \leq t_1$ . We will show that

$$\mathbf{Var}(Z_{\mathcal{N}}(t)) = O(r^\omega \mathbf{E}(Z_{\mathcal{N}}(t))) + e^{-a\omega} \mathbf{E}(Z_{\mathcal{N}}(t))^2. \quad (32)$$

for some constant  $a > 0$ .

Fix  $t \leq t_1$  and let  $\mathcal{E}_v$  be the event that vertex  $v \in \mathcal{R}(t)$ . Let  $\omega$  be as claimed just before Theorem 3. We claim that if  $v, w$  are at distance at least  $\omega$  then

$$\Pr(\mathcal{E}_v \cap \mathcal{E}_w) = (1 + e^{-\Omega(\omega)}) \Pr(\mathcal{E}_v) \Pr(\mathcal{E}_w). \quad (33)$$

We use Lemma 11. Let  $S = \{v, w\}$ . We argue that for the random walk on the associated  $H$  we have

$$R_\gamma = \rho + e^{-\Omega(\omega)} \text{ and } \pi_\gamma = \frac{2}{n} \text{ and hence } p_\gamma = (1 + O(Tr^{-\omega})) \frac{2}{\rho n}. \quad (34)$$

But the expression for  $p_\gamma$  is clear and the expression for  $R_\gamma$  can be demonstrated using the proof of Lemma 10. In this calculation one has to estimate the expected number of returns from  $v$  to  $v$ ,  $w$  to  $w$  and visits from  $v$  to  $w$  and vice-versa during the mixing time. The latter contributes  $O(Te^{-\Omega(\omega)}) = O(e^{-\Omega(\omega)}) \leq e^{-a\omega}$  because  $v$  and  $w$  are at distance at least  $\omega$ .

Equation (33) follows from Lemmas 8, 11 and (34). Thus,

$$\begin{aligned} \mathbf{E}(Z_{\mathcal{N}}^2(t)) &= \mathbf{E}(Z_{\mathcal{N}}(t)) + \sum_{v,w: \text{dist}(v,w) \geq \omega} \Pr(\mathcal{E}_v \cap \mathcal{E}_w) + \sum_{v,w: \text{dist}(v,w) < \omega} \Pr(\mathcal{E}_v \cap \mathcal{E}_w) \\ &\leq \mathbf{E}(Z_{\mathcal{N}}(t)) + (1 + e^{-a\omega})\mathbf{E}(Z_{\mathcal{N}}(t))^2 + r^\omega \mathbf{E}(Z_{\mathcal{N}}(t)) \end{aligned}$$

and (32) follows.

Applying the Chebyshev inequality we see that

$$\Pr(|Z_{\mathcal{N}}(t) - \mathbf{E}(Z_{\mathcal{N}}(t))| \geq \mathbf{E}(Z_{\mathcal{N}}(t))e^{-a\omega/3}) \leq \frac{2r^\omega e^{a\omega}}{\mathbf{E}(Z_{\mathcal{N}}(t))} + e^{-a\omega/3} = o(1)$$

for  $t \leq t_2$ .

This completes the proof of part (a) of Theorem 3.

The proof of part (b) is similar to that of (a). Observe first that Lemma 12 implies

$$\mathbf{E}(D_s(t)) = (1 + o(1))N_t \binom{r}{s} p_t^s (1 - p_t)^{r-s}. \quad (35)$$

It follows from (35) that

$$\mathbf{E}(D_s(t)) = \Omega(r^{\omega/\theta_{r,s}}) \text{ for } t \leq t_{s+1} - \omega n$$

We can use the Chebyshev inequality to prove concentration. We let  $\mathcal{F}_v(s)$  be the event that  $v$  is a vertex of degree  $s$  in  $\Gamma(t)$ . We prove that

$$\Pr(\mathcal{F}_v(s) \cap \mathcal{F}_w(s)) = (1 + e^{-\Omega(\omega)})\Pr(\mathcal{F}_v(s))\Pr(\mathcal{F}_w(s))$$

whenever  $v, w$  are at least  $\omega$  apart. We can argue for this by a small change in the argument for (a). This proves concentration for  $t \leq t_{s+1} - \omega n$  and proves part (b).  $\square$

### 4.3 Proof of Theorem 2

We combine Lemma 6 and Theorem 3 with the results of Molloy and Reed [16, 17]. We summarise what we need from the first two papers:

**Theorem 13.** Let  $\lambda_0, \lambda_1, \dots, \lambda_r \in [0, 1]$  be such that  $\lambda_0 + \lambda_1 + \dots + \lambda_r = 1$ . Suppose that  $\mathbf{d} = d_1, d_2, \dots, d_N$  satisfies  $|\{j : d_j = s\}| = (1 + o(1))\lambda_s N$  for  $s = 0, 1, \dots, r$ . Let  $G_{n, \mathbf{d}}$  be chosen randomly from graphs with vertex set  $[N]$  and degree sequence  $\mathbf{d}$ . Let

$$L = \sum_{s=1}^r s(s-2)\lambda_s.$$

- (a) If  $L < 0$  then **whp**  $G_{n, \mathbf{d}}$  is sub-critical.
- (b) If  $L > 0$  then **whp**  $G_{n, \mathbf{d}}$  is super-critical. Furthermore the unique giant component has size  $\theta n$  where  $\theta$  is defined as follows: Let  $\Lambda = \sum_{s=1}^r s\lambda_s$ . Define  $\alpha$  to be the smallest positive solution to

$$\Lambda - 2\alpha - \sum_{s=1}^r s\lambda_s \left(1 - \frac{2\alpha}{\Lambda}\right)^{s/2} = 0.$$

Then

$$\theta = 1 - \sum_{s=1}^r \lambda_s \left(1 - \frac{2\alpha}{\Lambda}\right)^{1/2}.$$

□

We now evaluate  $\Lambda$  in the context of  $\Gamma(t)$ . Then Theorem 3 implies that we can take

$$L = \sum_{s=1}^r \binom{r}{s} p_t^s (1-p_t)^{r-s} s(s-2) = r p_t ((r-1)p_t - 1).$$

Thus the critical value for  $t$  is the one that gives  $p_t = \frac{1}{r-1}$ . One can easily check that this is indeed the case for  $t^*$  as defined in (1). parts (i) and (ii) of Theorem 1 follow immediately.

To prove (iii) we use the result of Hatami and Molloy [13] that when  $|L| = O(n^{-1/3})$  the size of the giant is  $n^{2/3+o(1)}$ . At each step,  $L$  changes by  $O(1/n)$  and so at some time  $t \sim t^*$  the conditions of [13] will be satisfied. At this point **whp** there are  $\Theta(n)$  vertices in  $\Gamma(t)$  and (iii) follows. □

## 4.4 Proof of Theorem 4

In this section we study the number of components of a given size. In principle one should be able to work this out from Lemma 6 and Theorem 3. This has proven more difficult than we anticipated. Instead, we try to estimate the number directly. We can use these lemmas though to argue that almost all small components are trees. Indeed if we fix  $t$  and condition on the values  $D_s = D_s(t)$  satisfying Theorem 3 then we have the following:

**Lemma 14.**

(a) If  $t \leq \epsilon_1 n \log n$  then **whp** there are at most  $n^{3\epsilon_1}$  components of size  $k \leq \epsilon_1 \log n$  that are not trees.

(b) If  $t \geq \epsilon_1 n \log n$  then **whp** there are no components of size  $k \leq \log^2 n$  that are not trees.

**Proof** Let  $N = |\mathcal{R}(t)|$ . Applying Lemma 6 we see that the expected number of sets of  $k$  vertices that contain at least  $k$  edges is bounded by

$$\binom{N}{k} \binom{\binom{k}{2}}{k} \left(\frac{r}{n}\right)^k \leq \left(\frac{rNe^2}{2n}\right)^k.$$

To prove (a) we take  $N = n$  and apply the Markov inequality. To prove (b) we take  $N = N_t$ .  $\square$

With this in mind we concentrate on the number of tree components of size  $k$  for some  $k \leq \epsilon_1 \log n$ . Since there are **whp** at most  $n^{2\epsilon_1}$  vertices that are not nice, we will concentrate on counting the number of components that are made up of nice vertices only. We will also assume that  $t \leq t_1$ , see (31).

The following is proved in [2] – Lemma 4.

**Lemma 15.** *Let  $b_k$  be the number of subtrees of size  $k$  rooted at a vertex  $v$  in an infinite  $r$ -regular tree  $\mathcal{T}$ . Then*

$$b_k = \frac{r}{(r-2)k+2} \binom{(r-1)k}{k-1}.$$

$\square$

Now consider the situation described in Lemma 7. Fix  $v \in \mathcal{R}(t)$  and consider the  $k$  neighbourhood of  $v$  in the multi-graph on  $[n]$  induced by a random pairing on  $R(t)$ . It is **whp** a tree. Now delete any edge that corresponds to an edge  $(x, y)$  with  $x \in \mathcal{R}(t), y \in \mathcal{B}(t)$ . Let  $T$  be the component that contains  $v$ . If  $T$  has  $k$  vertices then  $T$  corresponds to a tree component of  $\Gamma(t)$  with  $k$  vertices.

So, fix a tree  $T^*$  as counted in Lemma 15 and let us determine the probability that  $T = T^*$ . The total degree of  $\mathcal{B}(t)$  is  $(1 + o(1))rN_t$  and it follows from Theorem 3 that the number of  $\mathcal{R}(t) : \mathcal{B}(t)$  edges is  $(1 + o(1))N_t r(1 - p_t)$ . So if we choose  $x \in W_{\mathcal{R}(t)}$  then it is paired with something in  $W_{\mathcal{R}(t)}$  with probability  $(1 + o(1))p_t$ , where the  $o(1)$  term is  $o(1/\log n)$ .

It follows that

$$\Pr(T = T^*) = (1 + o(1))p_t^{k-1}(1 - p_t)^{(r-2)k+2}. \quad (36)$$

To see this, start at the root  $v$  and examine the points paired with  $W_v$ . Each point will have probability  $(1 + o(1))p_t$  of being paired with an element of  $W_{\mathcal{R}(t)}$  and the  $o(1)$  term will not change by more than  $O(\log n/N_t) = o(n^{-1/5})$  as we proceed with this argument. The count

in Lemma 15 assumes an ordering of the neighbours of each vertex and by implication an ordering of  $W_v$  and a statement about which members of  $W_v$  are paired with  $W_{\mathcal{R}(t)}$  and which should not. Suppose we pair  $W_v$  with points from  $W_{x_i}, i = 1, 2, \dots, d$ . Then we continue by pairing up  $W_{x_1}$  and then  $W_{x_2}$  and so on. The factor  $p_t^{k-1}$  is from the  $k-1$  times we have to pair with  $W_{\mathcal{R}(t)}$  and the factor  $(1-p_t)^{(r-2)k+2}$  is from the number of times we do not. Note that the sizes of unpaired  $W_{\mathcal{R}(t)}$  and  $W_{\mathcal{R},\mathcal{B}(t)}(t)$  change by  $O(\log n)$  as we proceed.

It follows from (36) that

$$\mathbf{E}(N(k, t)) = (1 + o(1)) N_t \frac{b_k}{k} p_t^{k-1} (1 - p_t)^{(r-2)k+2}.$$

It remains to prove concentration around the mean. We use the Chebyshev inequality. We fix two vertex disjoint trees  $T_1, T_2$  in  $G$ . Arguing as above we see that

$$\Pr(T_1, T_2 \text{ are components of } \Gamma(t)) \leq (1 + o(1)) \prod_{i=1}^2 \Pr(T_i \text{ is a component of } \Gamma(t)).$$

We get what we want, provided  $\mathbf{E}(N(k, t)) \rightarrow \infty$  and it does so for  $t \leq (1 - \epsilon)t_k$ .

## 4.5 In the beginning

Using Lemma 6 we see that for  $1 \leq t \leq \log^3 n$  we have that  $\Gamma(t)$  is a random graph with a degree sequence of the following form: There are  $n - s$  vertices of degree  $r$ , where  $s \leq rt$ , and  $s$  vertices of degree  $< r$ . If the minimum degree in  $\Gamma(t)$  is at least one then **whp** we find that  $\Gamma(t)$  is connected. Indeed, let  $V_r$  be the set of vertices of degree  $r$  in  $\Gamma(t)$ . We argue that **whp**

$$V_r \text{ induces a connected subgraph of } \Gamma(t). \tag{37}$$

$$\text{Each } x \in \mathcal{R}(t) \setminus V_r \text{ is adjacent to } V_r. \tag{38}$$

For  $k$  even let

$$\phi(k) = \frac{k!}{(k/2)! 2^{k/2}}$$

be the number of ways of partitioning  $[k]$  into  $k/2$  pairs.

Let  $m = O(\log n)$  be the sum of the degrees, in  $\Gamma(t)$ , of the vertices in  $\mathcal{R}(t) \setminus V_r$ . Then

$$\begin{aligned}
\Pr((37) \text{ fails}) &\leq \sum_{k=3}^{n/2} \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \frac{\phi(kr+l)\phi((n-s-k)r+m-l)}{\phi((n-s)r+m)} \quad (39) \\
&= \sum_{k=3}^{n/2} \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \frac{\binom{(r(n-s)+m)/2}{(kr+l)/2}}{\binom{r(n-s)+m}{kr+l}} \\
&\leq \sum_{k=3}^{n/2} \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \frac{1}{\binom{(r(n-s)+m)/2}{(kr+l)/2}} \\
&\leq \sum_{k=3}^{n/2} \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \left( \frac{kr+l}{r(n-s)+m} \right)^{(kr+l)/2} \\
&= o(1).
\end{aligned}$$

**Explanation of (39):** Choose a set of  $k$  vertices  $S$  of degree  $r$  and  $l$  points from the  $m$  points in  $W$  associated with vertices  $T$  of degree less than  $r$ . (Some of the points associated with these latter vertices have already been paired). Now pair up the  $kr+l$  points randomly and the remaining points randomly. The  $l$  points contain the edges between  $S$  and  $T$ .

The probability that (38) fails is  $O(\log n/n)$ . A vertex of  $\mathcal{R}(t) \setminus V_r$  of degree  $d$  has an  $O(n^{-d})$  chance of not being connected to  $V_r$ .

So we only have to deal with the possibility that there are isolated vertices in  $\Gamma(t)$  for  $t \leq \log^3 n$ . So consider the event

$$\mathcal{A}(t) = \{\exists v \in N(W_t) : v \in \mathcal{R}(t) \text{ and } N(v) \subseteq \mathcal{B}(t)\}.$$

We claim that

$$\Pr(\mathcal{A}(t)) = O\left(\frac{\log^3 n}{(r-1)^{\ell_1/2}}\right). \quad (40)$$

It follows that

$$\Pr((38) \text{ fails}) \leq \sum_{t=1}^{\log^3 n} \Pr(\mathcal{A}(t)) = o(1).$$

To prove (40) fix  $t$  and a neighbour  $v$  of  $W_t$ . Equation (17) implies that there is at least one neighbour  $w$  of  $v$  that is not contained in a small cycle. If  $w \neq X_t$  then to reach  $w$  the walk  $\mathcal{W}$  must emulate a walk on the infinite tree  $\mathcal{T}$  that starts at distance  $\ell_1/2$  from the root and visits it. This has probability  $1/(r-1)^{\ell_1/2}$  and this must be inflated by  $\log^3 n$  to account for  $\log^3 n$  possible starting times. If  $w = X_t$  then to visit another neighbour of  $v$  then we must first reach distance at least  $\ell_1/2$  and then we can repeat the argument and use inequality (24).

## 5 The evolution of $\Gamma(t)$ in $G_{n,p}$

Because  $\Gamma(t)$  has the distribution  $G_{N,p}$ , we only need to get good estimates of  $N$  and we can get these from Lemma 8. Assume that  $c = n^{1/\omega}$  where  $\omega \rightarrow \infty$ . It is shown in [8] that **whp**  $R_v = 1 + O(1/\log n)$  for all  $v \in V$ . Let  $\omega_1 = \log^{1/3} n$ . For a fixed vertex  $v$  we have that its degree  $d_G(v)$  satisfies

$$\Pr(|d_G(v) - c \log n| \geq \omega_1 (c \log n)^{1/2}) \leq 2e^{-\omega_1^2/3}.$$

This follows from Chernoff bounds on the binomial.

So by the Markov inequality, we see that **whp** all but  $O(ne^{-\omega_1^2/4})$  vertices have degrees in the range  $c \log n \pm \omega_1 (c \log n)^{1/2}$ . Denote these vertices by  $\mathcal{N}_d$ . Fix such a vertex, then (15) implies that

$$\Pr(v \in \mathcal{R}(t)) = (1 + O(\log n/n))e^{-(1+O(1/\omega_1))t/n} = e^{-(1+O(1/\omega_1))t/n}$$

if we assume that  $t \sim n \log \log n$ . So if  $t = t_\theta$  where  $\theta = O(1)$  then

$$\mathbf{E}(|\mathcal{N}_d|) = (1 + o(1)) \frac{n}{c^{1+\theta} \log n}.$$

We can argue as in the proof (33) that if  $v, w \in \mathcal{N}_d$  are at distance at least  $\omega/2$  in  $G$  then

$$\Pr(\mathcal{E}_v \cap \mathcal{E}_w) = (1 + o(1))\Pr(\mathcal{E}_v)\Pr(\mathcal{E}_w). \quad (41)$$

where  $\mathcal{E}_v = \{v \in \mathcal{R}(t)\}$ .

Then if  $X = |\mathcal{R}(t) \cap \mathcal{N}_d|$  then

$$\mathbf{E}(X(X-1)) \leq \mathbf{E}(X)^2 + O((c \log n)^{\omega/2})\mathbf{E}(X)$$

which implies that  $\mathbf{Var}(X) = o(\mathbf{E}(X)^2)$  and then the Chebyshev inequality implies that  $X \sim \mathbf{E}(X) \sim \frac{n}{c^{1+\theta} \log n}$  **whp**. The vertices outside  $\mathcal{N}_d$  only contribute  $o(N)$  **whp** and Theorem 1 follows immediately from this.

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