

Hamilton Cycles in Random Lifts of Graphs

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Abstract

An n -lift of a graph K , is a graph with vertex set $V(K) \times [n]$ and for each edge $(i, j) \in E(K)$ there is a perfect matching between $\{i\} \times [n]$ and $\{j\} \times [n]$. If these matchings are chosen independently and uniformly at random then we say that we have a random n -lift. We show that there are constants h_1, h_2 such that if $h \geq h_1$ then a random n -lift of the complete graph K_h is hamiltonian **whp** and if $h \geq h_2$ then a random n -lift of the complete bipartite graph $K_{h,h}$ is hamiltonian **whp**.

1 Introduction

For a graph K , an n -lift G of K has vertex set $V(K) \times [n]$ where for each vertex $v \in V(K)$, $\{v\} \times [n]$ is called the *pillar* above v and will be denoted by Π_v . The edge set of a an n -lift G consists of a perfect matching between pillars Π_u and Π_w for each edge $(u, w) \in E(K)$. The set of n -lifts will be denoted $\mathcal{L}_n(K)$. In this paper we discuss random n -lifts, chosen uniformly from $\mathcal{L}_n(K)$. In this case, the matchings between pillars are chosen independently and uniformly at random.

Lifts of graphs were introduced by Amit and Linial in [1] where they proved that if K is a connected, simple graph with minimum degree $\delta \geq 3$, and G is chosen randomly from $\mathcal{L}_n(K)$ then G is δ -connected **whp**, where the asymptotics are for $n \rightarrow \infty$. They continued the study of random lifts in [2] where they proved expansion properties of lifts. Together with Matoušek, they gave bounds on the independence number and chromatic number of random lifts in [3]. Linial and Rozenman [4] give a tight analysis for when a random n -lift has a perfect matching.

In this paper we discuss the probability that a random n -lift is hamiltonian. In particular we study the case where K is the complete graph K_h or the complete bipartite graph $K_{h,h}$. We use the notation $y \stackrel{r}{\in} Y$ for “ y is chosen uniformly at random from Y ”.

Theorem 1. *There exists a constant h_1 such that if $h \geq h_1$ and $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$ then G is hamiltonian **whp**.*

Theorem 2. *There exists a constant h_2 such that if $h \geq h_2$ and $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$ then G is hamiltonian **whp**.*

Theorem 1 is proved in the next section. Theorem 2 is proved in Section 3.

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2 Proof of Theorem 1

2.1 Structural Properties of $\mathcal{L}_n(K_h)$

The vertices of $\mathcal{L}_n(K_h)$ will be denoted by V and its edges will be denoted E .

We will use the coloring argument of Fenner and Frieze [7] to show G is hamiltonian **whp**. For $G \in \mathcal{L}_n(K_h)$ we choose a set $H_1 = H_1(G) \subseteq E(G)$ as follows: Each vertex of G arbitrarily chooses 12 edges of G incident with it. Thus the number of distinct edges chosen is between $6hn$ and $12hn$ and the minimum degree of the graph induced by H_1 is at least 12. Next let $P_0 = P_0(G)$ be a specific longest path in G . Let $F(G) = P_0 \cup H_1$ be the fixed edges of G .

The analysis uses an *unspecified*, sufficiently small, positive constant $\beta < 1$.

Let $\mathcal{B} = \mathcal{B}(G)$ be the set of subsets of $E(G)$ of size $\beta \binom{h}{2} n$. We say that a subset of edges H is *acceptable* if $H = B \cup F$ for some $B \in \mathcal{B}(G)$. Let $\mathcal{H}(G)$ be the collection of acceptable subgraphs of G . For a lift G , each $B \in \mathcal{B}(G)$ defines a coloring of the edges of G in which the edges of $H = B \cup F$ are colored blue and the edges of $R = G \setminus H$ are colored red.

Let $S \subseteq V$ be of size s and let S_i be the intersection of $S \subseteq V$ with pillar Π_i for $i \in [h]$. The number of choices for S is $\binom{hn}{s}$ and by considering the number of choices for the S_i we see that

$$\sum_{s_1 + \dots + s_h = s} \prod_i \binom{n}{s_i} = \binom{hn}{s} \leq \left(\frac{hne}{s}\right)^s. \quad (1)$$

For a graph $G = (V, E)$ and $S \subseteq V$ let $N(S) = \{v \in V \setminus S : \exists u \in S \text{ such that } (u, v) \in E(G)\}$ be the disjoint neighborhood of S .

For $G \in \mathcal{L}_n(K_h)$ and sets $S \subseteq \Pi_i$ and $T \subseteq \Pi_j$, $|S| = s, |T| = t$,

$$\Pr(N(S) \cap \Pi_j \subseteq T) = \frac{t(t-1)\dots(t-s+1)}{n(n-1)\dots(n-s+1)} \leq \left(\frac{t}{n}\right)^s. \quad (2)$$

Throughout this section all statements hold for n and h sufficiently large.

Lemma 1. For $G \in \mathcal{L}_n(K_h)$,

$$\Pr(\exists S \subseteq V : |S| \leq \frac{n}{10h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}) = o(1)$$

Proof Using (1) we see that the expected number of sets S of size s that contain at least $2s$ edges is no more than

$$\begin{aligned} \phi(s) &= \sum_{s_1 + \dots + s_h = s} \prod_i \binom{n}{s_i} \binom{\binom{s}{2}}{\binom{2s}{2}} \left(\frac{1}{n-2s}\right)^{2s} \\ &\leq \binom{hn}{s} \left(\frac{s^2 e}{4s}\right)^{2s} \left(\frac{1}{n(1-\frac{1}{5h})}\right)^{2s} \\ &\leq \left(\frac{hne}{s}\right)^s \left(\frac{se}{4}\right)^{2s} \left(\frac{2}{n}\right)^{2s} \\ &\leq \left(\frac{e^3 hs}{4n}\right)^s \end{aligned}$$

Then

$$\sum_{s=5}^{n/10h} \phi(s) = o(1).$$

□

Lemma 2. *If $G \in \mathcal{L}_n(K_h)$ and $H \in \mathcal{H}(G)$, then **whp** H satisfies*

$$S \subseteq V, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \quad (3)$$

Proof Assume first that $|S| \leq n/10h$ and let $U = S \cup N(S)$. Let a be the number of edges contained in S and let b be the number of edges from S to $N(S)$. The degree sum of S in H_1 is at least $12|S|$ and so $2a + b \geq 12|S|$. But then U contains at least $a + b \geq 6|S|$ edges and we can assume by Lemma 1 that $|U| > 3|S|$. This completes the argument for $|S| \leq n/10h$.

Let H' be defined by including an edge of G in H' independently with probability β' where $\beta' < \beta$. Then $|H'|$ is a binomial random variable whose expected value is less than $\beta \binom{h}{2} n$. The Chernoff bound implies that for a monotone increasing property of lifts \mathcal{Q} , if $H' \in \mathcal{Q}$ **whp**, then $H \in \mathcal{Q}$ **whp**.

For $n/10h < |S| \leq hn/4$, let $T = N(S)$ and $t = |T|$. Using (1) and (2), the expected number Z of sets S with $|N_{H'}(S)| < 2|S|$ is bounded as follows: In the first line of the following display, the notation $j \succ i$ denotes $s_j + t_j > s_i + t_i$ or $s_j + t_j = s_i + t_i$ and $j > i$.

$$\begin{aligned} Z &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \sum_{t_1+\dots+t_h=t} \prod_i \binom{n}{s_i} \prod_j \binom{n}{t_j} \prod_{i=1}^h \prod_{j \succ i} \left(\beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{s_i + t_i} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \prod_{i=1}^h \prod_{j \neq i} \left(\beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{(s_i + t_i)/2} \\ &= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \prod_{j=1}^h \left(\beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{(s+t - (s_j + t_j))/2} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \left(\sum_{j=1}^h \left(\beta' \frac{s_j + t_j}{(h-1)n} + (1 - \beta') \right) \right)^{(h-1)(s+t)/2} \\ &= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \left(\beta' \frac{s+t}{(h-1)n} + (1 - \beta') \right)^{(h-1)(s+t)/2} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left(\frac{neh}{s} \right)^s \left(\frac{neh}{t} \right)^t \left(1 - \beta' \left(1 - \frac{s+t}{(h-1)n} \right) \right)^{(h-1)(s+t)/2} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left(\frac{neh}{s} \right)^s \left(\frac{neh}{t} \right)^t \exp \left\{ -\beta' \left(1 - \frac{s+t}{(h-1)n} \right) (h-1)(s+t)/2 \right\} \\ &\leq \sum_{s=n/10h}^{hn/4} \left(\frac{neh}{s} \right)^{3s} \exp \left\{ -\frac{\beta'hs}{10} \right\} \\ &\leq e^{-\beta n/199}. \end{aligned} \quad (4)$$

□

Lemma 3. *If $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$ and $H \stackrel{r}{\in} \mathcal{H}(G)$, then **whp** H is connected.*

Proof If H is not connected, Lemma 2 implies that **whp** H is the union of a constant number of components of size at least $hn/4$. We will again work under the assumption that edges are included in H' independently with probability β' where $\beta' < \beta$.

Assume without loss of generality that $|S| \leq hn/2$. The expected number of sets S of size $|S| \in [hn/4, hn/2]$ with no edges between S and its complement is no more than

$$\begin{aligned}
& \sum_{s=hn/4}^{hn/2} \left(\sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \right) \prod_{i=1}^h \prod_{j>i} \left(\beta' \left(\frac{s_j}{n} \right) + (1-\beta') \right)^{s_i} \\
\leq & \sum_{s=hn/4}^{hn/2} \left(\frac{neh}{s} \right)^s \left(\beta' \left(\frac{s}{(h-1)n} \right) + (1-\beta') \right)^{(h-1)s/2} \\
\leq & \sum_{s=hn/4}^{hn/2} \left(\frac{neh}{s} \right)^s \exp \left\{ -\frac{\beta' s}{2} (h/2 - 1) \right\} \\
\leq & e^{-\beta h^2 n/5}
\end{aligned} \tag{5}$$

□

Let $P = (v_0, \dots, v_k)$ be a longest path in graph H . A *Pósa rotation* of P [10] with v_0 fixed gives another longest path $P' = (v_0, \dots, v_i v_k \dots v_{i+1})$ created by adding edge (v_k, v_i) and deleting edge (v_i, v_{i+1}) . Let $END_H(v_0, P)$ be the set of endpoints obtained by a sequence of Pósa rotations starting with P , keeping v_0 fixed and using an edge (v_k, v_i) of H .

Each vertex $v_j \in END_H(v_0, P)$ can then be used as the initial vertex of another set of longest paths $END_H(v_j, P)$, this time using v_j as the fixed vertex, but again only adding edges from H . Let $END_H(P) = \{v_0\} \cup END_H(v_0, P)$.

The Pósa condition

$$|N(END(v, P))| \leq 2|END(v, P)| - 1$$

for $v \in END_H(P)$ together with Lemma 2 implies the following.

Lemma 4. *If $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$ and $H \stackrel{r}{\in} \mathcal{H}(G)$, then **whp** $|END_H(v, P)| \geq hn/4$ for all $v \in END_H(P)$, $P = P(G)$.*

We say next that an *ordered* pair of pillars (Π_k, Π_l) is *good* w.r.t. a longest path P if

$$|\{u \in \Pi_k \cap END_H(P) : |\{v \in \Pi_l \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/500\}| \geq n/500. \tag{6}$$

In words, Π_k contains at least $n/500$ vertices $u \in END_H(P)$ for which there at least $n/500$ vertices $v \in \Pi_l \cap END_H(u, P)$ such that the edge $(u, v) \notin E(H)$.

Lemma 5. *If (3) holds then G has at least $\binom{h}{2}/3000$ good pillar pairs.*

Proof We show first that for $u \in END_H$ there are at least $h/7 - 1$ pillars for which

$$|\{v \in \Pi_l \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/8 \tag{7}$$

holds. Let $u \in \text{END}_H$ and suppose that there are m pillars for which (7) fails. The total number of vertices in $\text{END}(u, H)$ must be at least $hn/4$ by Lemma 4 which gives the inequality

$$mn/8 + (h - m)n \geq hn/4$$

so that $m \leq 6h/7$. We get $h/7 - 1$ “good” pillars, because we have to discount the pillar containing u .

Next, we say that a non-edge $(x, y) \notin E(H)$ must be *avoided* if $x \in \text{END}_H$ and $y \in \text{END}(u, H)$. We have just shown that for each $u \in \text{END}_H$, there are at least $hn/57$ edges incident with u that must be avoided. As $|\text{END}_H| \geq |\text{END}(u, H)|$ and each non-edge is counted at most twice, the total number of non-edges in G that must be avoided is at least $\frac{1}{2}hn/4 \cdot hn/57$.

Assume now that there are $\delta \binom{h}{2}$ pillar pairs that contain at least $n^2/250$ edges that must be avoided. We then get the inequality

$$\delta \binom{h}{2} n^2 + (1 - \delta) \binom{h}{2} n^2 / 250 \geq h^2 n^2 / 456$$

which gives $\delta > 1/3000$.

Let (Π_k, Π_l) be a pillar pair that contains at least $n^2/250$ edges that must be avoided. To show that (Π_k, Π_l) is good, let

$$|\{u \in \text{END} \cap \Pi_k : |\{v \in \text{END}(u, H) \cap \Pi_l, (u, v) \notin E(H)\}| \geq n/500\}| = \gamma n. \quad (8)$$

We then get the inequality

$$\gamma n^2 + (1 - \gamma)n^2/500 \geq n^2/250$$

so $\gamma > 1/500$. □

2.2 The Proof

For a lift G , let $\mathcal{D}(G)$ be the subset of $\mathcal{H}(G)$ in which H is connected and satisfies (3) for $|S| > n/10h$ and let $\mathcal{D} = \cup_G \mathcal{D}(G)$. Let \mathcal{A} be the subset of $\mathcal{L}_n(K_h)$ such that for $G \in \mathcal{A}$ and H chosen randomly from $\mathcal{H}(G)$,

$$\Pr(H \in \mathcal{D}(G)) \geq 1 - \alpha.$$

where $\alpha = e^{-\beta n/200}$.

Let \mathcal{C} be the subset of $\mathcal{L}_n(K_h)$ that is not hamiltonian and let $\mathcal{F} = \mathcal{A} \cap \mathcal{C}$. To show that $\Pr(\mathcal{C}) \rightarrow 0$, we will first show that $|\mathcal{A}| = (1 - o(1)) |\mathcal{L}_n(K_h)|$ and then use the coloring argument of Fenner and Frieze [7] to show that $\Pr(\mathcal{F}) \rightarrow 0$.

Lemma 6. $|\mathcal{A}| = (1 - o(1)) |\mathcal{L}_n(K_h)|$

Proof If $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$ and $H \stackrel{r}{\in} \mathcal{H}(G)$ then

$$\begin{aligned} \Pr(H \in \mathcal{D}) &= \sum_{G \in \mathcal{L}_n(K_h)} \Pr(H \in \mathcal{D}|G) \Pr(G) \\ &= \sum_{G \in \mathcal{A}} \Pr(H \in \mathcal{D}|G) \Pr(G) + \sum_{G \notin \mathcal{A}} \Pr(H \in \mathcal{D}|G) \Pr(G) \\ &\leq \Pr(\mathcal{A}) + (1 - \alpha)(1 - \Pr(\mathcal{A})) \\ &= 1 - \alpha + \alpha \Pr(\mathcal{A}) \end{aligned} \quad (9)$$

and (4) and (5) imply that

$$\Pr(H \in \mathcal{D}) \geq 1 - \alpha^2. \quad (10)$$

Putting (9) and (10) together, we get

$$1 - \alpha + \alpha \Pr(\mathcal{A}) \geq 1 - \alpha^2.$$

so that

$$\Pr(\mathcal{A}) \geq 1 - \alpha. \quad \square$$

To get an upper bound on the number of graphs $G \in \mathcal{L}_n(K_h)$ such that $G \in \mathcal{F}$, we construct a 0-1 matrix $A = \|a_{i,j}\|$. Row index i corresponds to a graph $G_i \in \mathcal{L}_n(K_h)$ and index j ranges over all acceptable subgraphs $H \in \mathcal{H}(G_i)$. Subgraph j of G_i will be denoted by $H_{i,j}$. Let

$$a_{i,j} = 1 \text{ if } \begin{cases} (i) & S \subseteq V, |S| \leq hn/4 \text{ implies } |N_{H_{i,j}}(S)| \geq 2|S| \\ (ii) & H_{i,j} \text{ is connected} \\ (iii) & H_{i,j} \supseteq P_0(G_i) \\ (iv) & G_i \text{ is not Hamiltonian} \\ (v) & |E_{H_{i,j}}(\Pi_k, \Pi_l)| \in [(1 \pm n^{-1/3})\beta n], \forall k \neq l \in [h] \end{cases} \quad (11)$$

Note that (ii), (iii) and (iv) imply

$$\exists \text{ longest path } P \text{ of } H_{i,j}, (u, v) \in E(R_{i,j}) : u \in \text{END}_{H_{i,j}}(P), v \in \text{END}_{H_{i,j}}(u, P) \quad (12)$$

Now let

$$N_1 = \sum_i \sum_j a_{i,j}$$

be the number of ones in A .

Lemma 7. *If $G_i \in \mathcal{F}$ then*

$$\sum_j a_{i,j} \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1 - \beta)\binom{h}{2}n - 13hn}.$$

Proof $G_i \in \mathcal{F}$ and $H_{i,j} \in \mathcal{H}(G_i)$ implies that $H_{i,j}$ satisfies (i), (ii), (iii) and (iv) **whp**. Now $B_1, B_2 \in \mathcal{B}(G)$ may give rise to the same subgraph H if the edges not in $B_1 \cap B_2$ are all in F . So we count the number of ways to select R as a lower bound on $|\mathcal{H}(G_i)|$. We have $|H| \leq \beta \binom{h}{2}n + 13hn$ since there are at most $13hn$ edges in P_0 and H_1 . Then the number of choices for R is at least the number of ways to select a set of $(1 - \beta)\binom{h}{2}n - 13hn$ edges from the $\binom{h}{2}n - 13hn$ not in F . Condition (v) holds through the Chernoff bound. \square

It follows immediately from Lemma 7 that

$$N_1 \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1 - \beta)\binom{h}{2}n - 13hn} |\mathcal{F}|. \quad (13)$$

We now obtain an upper bound on N_1 . Let

$$\mathcal{X} = \{H : \exists i, j \text{ for which } H_{i,j} = H \text{ and } a_{i,j} = 1\}$$

The following bound follows from the definition and a concentration inequality for sampling without replacement, see Hoeffding [9], Theorem 4:

$$|\mathcal{X}| \leq \binom{\binom{h}{2}n}{13hn} \left((1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{\binom{h}{2}}. \quad (14)$$

For a fixed $H \in \mathcal{X}$ let

$$\mathcal{G}_H = \{G_i : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H \in \mathcal{X}} |\mathcal{G}_H|.$$

Lemma 8.

$$H \in \mathcal{X} \text{ implies } |\mathcal{G}_H| \leq e^{-ch^2n} \left(((1 - \beta + O(n^{-1/3}))n)! \right)^{\binom{h}{2}} \quad (15)$$

for some absolute constant $c > 0$.

Proof We begin with H and count the number of ways to add back the edges of R to form a lift $G_i \in \mathcal{G}_H$. The number of edges in $R(k, l)$ between two pillars of G_i is no more than $(1 - \beta + O(n^{-1/3}))n$. Thus there are at most $((1 - \beta + O(n^{-1/3}))n)!$ possible matchings to add back between each pair of pillars.

When adding back new edges to H we must avoid edges (u, v) where $u \in \text{END}_H$ and $v \in \text{END}(u, H)$ so that $a_{i,j} = 1$ in the resulting graph. For a good pillar pair (Π_k, Π_l) as defined in (6), there are at least $n/500$ vertices $x \in \Pi_k$, each adjacent at least $n/500$ vertices $y \in \Pi_l$ that give rise to an edge (x, y) that must be avoided. The probability that we avoid all such edges between a good pillar pair is at most

$$\prod_{i=0}^{n/500-1} \left(1 - \frac{n/500 - i}{n - i} \right) \leq e^{-n/250,000}$$

As there are at least $\binom{h}{2}/3000$ good pillar pairs, the probability that a set of new edges avoids all required edges in G_i is at most $(e^{-n/250,000})^{\binom{h}{2}/3000}$. \square

It follows from (13), (14) and (15) that $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$ is bounded above by

$$\begin{aligned} & \frac{e^{-ch^2n} ((1 - \beta + O(n^{-1/3}))n)! \binom{h}{2} \binom{\binom{h}{2}n}{13hn} \left((1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{\binom{h}{2}}}{(1 - o(1))(n!) \binom{h}{2} \binom{\binom{h}{2}n - 13hn}{(1-\beta)\binom{h}{2}n - 13hn}} \\ & \leq \frac{e^{-ch^2n/2} \binom{n}{\beta n} \binom{h}{2}}{\binom{\binom{h}{2}n}{\beta \binom{h}{2}n} \beta^{14hn}} \\ & \leq e^{-ch^2n/2 + 14hn \ln(1/\beta)} \\ & = o(1) \end{aligned}$$

where the second line uses $\binom{a-x}{b-x} \geq \left(\frac{b-x}{a-x}\right)^x \binom{a}{b}$. \square

3 Proof of Theorem 2

3.1 Structural Properties of $\mathcal{L}_n(K_{h,h})$

Let V_1, V_2 be the bipartition of $K_{h,h}$ and let W_1, W_2 be the bipartition of the lifts of $K_{h,h}$ that it induces.

We now prove similar properties to those in Section 2.1. Let H_1, P_0 be sets of edges defined as in Section 2.1 and let $F = P_0 \cup H_1$. Again we use an unspecified, suitably small constant $\beta < 1$, let B be a set of $\beta \binom{h}{2} n$ edges in G and $\mathcal{B}(G)$ the collection of subgraphs B . A set of edges H in G is acceptable if $H = B \cup F$ for some $B \in \mathcal{B}(G)$. Let $\mathcal{H}(G)$ be the collection of acceptable subgraphs of G and let $R = G \setminus H$.

Throughout this section all statements hold for n and h sufficiently large. The proof is similar to that for K_h and so we will omit calculations that are almost identical to those of the previous sections.

The main difficulty with using a Posá type argument is that if a longest path P in G is even then it cannot be closed to a cycle, connectivity notwithstanding i.e. we gain nothing from avoiding choosing green edges to join v to $END(v)$. In this case, there are no edges to avoid. We therefore have to modify the argument. We follow Bollobás and Kohayakawa [6] who considerably simplified the argument of [8].

Lemma 9. For $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$

$$\Pr(\exists S \subseteq V : |S| \leq \frac{n}{20h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}) = o(1)$$

□

Lemma 10. If $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$ and $H \stackrel{r}{\in} \mathcal{H}(G)$, then **whp** H satisfies

$$S \subseteq W_i, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \quad (16)$$

□

Lemma 11. If $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$ and $H \stackrel{r}{\in} \mathcal{H}(G)$ then **whp** H is connected.

□

Lemma 12. If K has a 2-factor and $G \in \mathcal{L}_n(K)$, then G has a 2-factor.

Proof Let $C \subseteq V(K)$ be one of the cycles of a 2-factor of K and let $G[C]$ the subgraph of G induced by the pillars above the vertices of C . Let v_1, \dots, v_k be an ordering of the vertices of C such that (v_i, v_{i+1}) is an edge of C (where $v_1 = v_{k+1}$) and let Π_i be the pillar of G above $v_i \in C$. Let σ_i be the permutation that defines the matching from pillar Π_i to Π_{i+1} for each $\Pi_i \in G[C]$. For each $j \in \Pi_1$, define $\sigma(j) = \sigma_k \sigma_2 \cdots \sigma_1(j)$ to be the permutation on the vertices of Π_1 that results from following the permutations σ_1 through σ_k back to Π_1 . Then a cycle of σ is a cycle of G so that the cycles of σ define a 2-factor of $G[C]$. This process can be repeated for all cycles of a 2-factor of K to obtain a 2-factor of $G \in \mathcal{L}_n(K)$. □

We now describe an extension-rotation process which attempts to transform the 2-factor F of Lemma 12 into a Hamilton cycle.

General Step: Given the current 2-factor (initially F) choose an edge $e = (x, y)$ of G which joins two distinct cycles C, C' . This is possible because G is connected **whp**. Let f be an edge of

C incident with x and f' be an edge of C' incident with y . Let P be the path $C \cup C' \cup \{e\} \setminus \{f, f'\}$. There are now several possibilities.

(a): There is an endpoint u say, of P which has a neighbour v in a cycle C'' disjoint from P . We *extend* P by replacing P, C'' by $P \cup C'' \cup \{(u, v)\} \setminus f''$ where f'' is an edge of C'' incident with v . We repeat this operation as long as we can. We then carry out (b) or (c).

(b) The endpoints u, v of P are connected by an edge in H . Adding (u, v) to P creates a 2-factor with at least one less cycle than at the start of the General Step and completes it.

(c) Carry out rotations on P until either (i) we construct a path Q with an endpoint x which is adjacent to a vertex y on cycle C outside Q or (ii) we satisfy the condition of (b). In the latter case we proceed as in (b) above. In the former case we extend Q by adding the edge (x, y) and deleting an edge of C incident with y .

We continue the above operations until we either obtain a Hamilton cycle or obtain a path $P_0 = P_0(G) = (v_0, v_1, \dots, v_p)$ that cannot be extended or closed to a cycle via a sequence of rotations. Note that this path is necessarily of odd length.

We therefore let P_0 be a longest path of *odd* length which (i) cannot be extended by rotations and (ii) for which there are a set of vertex disjoint cycles covering the vertices not in P .

We use the Pósa condition (which still holds) and Lemma 10 to get the following.

Lemma 13. *If $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$ and $H \stackrel{r}{\in} \mathcal{H}(G)$, then **whp** $|END_H(v, P_0)| \geq hn/4$ for all $v \in END_H(P_0)$, $P_0 = P_0(G)$.*

We say next that an *ordered* pair of pillars (Π_k, Π_l) is *good* w.r.t. a longest path P if $\Pi_k \in W_x$, $\Pi_l \in W_{3-x}$, $x = 1, 2$ and

$$|\{u \in \Pi_k \cap END_H(P) : |\{v \in \Pi_l \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/500\}| \geq n/500. \quad (17)$$

In words, Π_k contains at least $n/500$ vertices $u \in END_H(P)$ for which there are at least $n/500$ vertices $v \in \Pi_l \cap END_H(u, P)$ such that the edge $(u, v) \notin E(H)$.

Lemma 14. *If (16) holds then G has at least $\binom{h}{2}/3000$ good pillar pairs.*

Proof We first note that P_0 and the paths obtained by rotations are of odd length and so each has one endpoint in each of W_1, W_2 .

Now we can argue as in Lemma 5 that for each $u \in W_x \cap END_H$, $x = 1, 2$ there are at least $h/7$ pillars $\Pi_j \in W_{3-x} \cap END(u, H)$ for which

$$|\{v \in \Pi_k \cap END_H(u, P) : (u, v) \notin E(H)\}| \geq n/8.$$

The rest of the proof is identical to that of Lemma 5. □

3.2 The Proof

Define the sets $\mathcal{A}, \mathcal{C}, \mathcal{F}$ as in the proof of Theorem 1. We have $|\mathcal{A}| \geq (1 - o(1)) |\mathcal{L}_n(K_{h,h})|$ using the argument in Lemma 6 with the results from Lemmas 10 and 11. Define also the matrix A and N_1 as in the proof of Theorem 1. The proofs of the following Lemmas are similar to the proofs of Lemmas 7 and 8.

Lemma 15. *If $G_i \in \mathcal{F}$ then*

$$\sum_j a_{i,j} \geq (1 - o(1)) \binom{h^2 n - 23hn}{(1 - \beta)h^2 n - 23hn}.$$

It follows immediately from Lemma 15 that

$$N_1 \geq (1 - o(1)) \binom{h^2 n - 23hn}{(1 - \beta)h^2 n - 23hn} |\mathcal{F}|. \quad (18)$$

We now obtain an upper bound on N_1 . Let

$$\mathcal{X} = \{H : \exists i, j \text{ for which } G_{i,j} = H \text{ and } a_{i,j} = 1\}$$

It follows from the definition that

$$|\mathcal{X}| \leq \binom{h^2 n}{23hn} \left((1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{h^2}. \quad (19)$$

For a fixed $H \in \mathcal{H}$ let

$$\mathcal{G}_H = \{G_{i,j} : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H \in \mathcal{X}} |\mathcal{G}_H|.$$

Lemma 16.

$$H \in \mathcal{X} \text{ implies } |\mathcal{G}_H| \leq e^{-ch^2 n} \left((1 - \beta + O(n^{-1/3})n)! \right)^{h^2}. \quad (20)$$

It follows from (18), (19) and (20) that $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$ is bounded above by

$$\begin{aligned} & \frac{e^{-ch^2 n} \left((1 - \beta + O(n^{-1/3})n)! \right)^{h^2} \binom{h^2 n}{23hn} \left((1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{h^2}}{(1 - o(1))(n!)^{h^2} \binom{h^2 n - 23hn}{(1 - \beta)h^2 n - 23hn}} \\ & \leq \frac{e^{-ch^2 n/2} \binom{n}{\beta n}^{h^2}}{\binom{h^2 n}{\beta h^2 n} \beta^{24hn}} \\ & \leq e^{-ch^2 n/2 + 24hn \ln(1/\beta)} \\ & = o(1). \end{aligned}$$

References

- [1] A. Amit and N. Linial, *Random Graph Coverings I: General Theory and Graph Connectivity*, *Combinatorica* 22 (2002), 1-18.
- [2] A. Amit and N. Linial, *Random Lifts of Graphs II: Edge Expansion*, *Combinatorics Probability and Computing*, to appear.

- [3] A. Amit, N. Linial and J. Matoušek, *Random Lifts of Graphs III: Independence and Chromatic Number*, Random Structures and Algorithms 20 (2002), 1-22.
- [4] N. Linial, and E. Rozenman, *Random Lifts of Graphs: Perfect Matchings*, to appear.
- [5] B. Bollobás, *Random Graphs*, Cambridge University Press (2001).
- [6] B. Bollobás and Y. Kohayakawa, *The hitting time of Hamilton cycles in random bipartite graphs*, Graph Theory, Combinatorics, Algorithms and Applications, (Y. Alavi, F.R.K. Chung, R. Graham and D.F. Hsu, Eds.) (1991) 26-41.
- [7] T.I. Fenner and A.M. Frieze, *On the existence of Hamilton cycles in a class of random graphs*, Discrete Mathematics 45 (1983), 301-305.
- [8] A.M. Frieze, *Limit distribution for the existence of hamiltonian cycles in random bipartite graphs*, European Journal of Combinatorics 6 (1985) 327-334.
- [9] W. Hoeffding, *Probability Inequalities for Sums of Bounded Random Variables*, American Statistical Association Journal (1963), 13-30.
- [10] L. Pósa, *Hamilton circuits in random graphs*, Discrete Mathematics 14 (1976), 359-364.