Distribution of vertex degree in web-graphs

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Abstract

We give results for the age dependent distribution of vertex degree and number of vertices of given degree in the undirected web-graph process, a discrete random graph process introduced in [8]. For such processes we show that as $k \to \infty$, the expected proportion of vertices of degree k has power law parameter $1+1/\eta$ where η is the limiting ratio of the expected number of edge endpoints inserted by preferential attachment to the expected total degree. The proof for the undirected process generalizes naturally to give similar results for the directed hub-authority process, and an undirected hypergraph process.

1 Introduction

Discrete random processes exhibiting power law properties have been studied by many authors and in many contexts since Yule [15]. Recent interest was stimulated by the papers of Barabási and Albert [1] who observed a power law degree sequence for a subgraph of the World Wide Web (WWW) and by Faloutsos, Faloutsos and Faloutsos [10] who observed power law behaviour for the internet graph. Further empirical studies giving power laws for a larger portion of the WWW were subsequently made by other researchers, in particular Broder et al. [6].

The central feature of processes with a power law degree sequence is that the *proportional degree sequence*, (the expected proportion $N_t(k)$ of vertices of degree k in the process at step t) is asymptotically independent of t, and exhibits a power law relationship. That is to say

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$N_t(k) \sim Ck^{-x}$ for a wide range of $k \to \infty$, where x is a positive constant depending only on the process.

Many discrete random processes have been devised to generate power law degree sequences. A typical example of such a process starts with an existing graph $G(0)$ and forms a sequence of graphs $(G(t))_{t\geq 0}$ by adding a new vertex v_t at each step t along with a m edges $e_{i,t} = (v_t, w_{i,t}), i = 1, ..., m$. The terminal vertex $w_{i,t}$ is chosen from the existing vertices by preferential attachment (with probability proportional to vertex degree). Preferential attachment gives a power law whereas choosing terminal vertices uniformly at random (uar) gives a degree sequence with geometric decay. This method was proposed by Barabási and Albert [1] as a generative procedure for a model of the WWW. Surveys by Bollobás and Riordan [3] and Drinea, Enachescu and Mitzenmacher [9] give many generative procedures to obtain graphs with power laws.

We briefly outline three models which give power laws; the scale-free process of Bollobás and Riordan, the web-graph process of Cooper and Frieze, which is the topic of this paper and the copying process of Kumar et al. In the simplest case the first two processes have proportional degree sequence with power law exponent $x = 3$.

The scale-free model of Bollobás and Riordan [2], [5] is based on a process $(G_1(t))_{t\geq 0}$ which adds a new vertex and a single new edge at each step. The terminal vertex of the new edge is chosen preferentially among all vertices, and the new vertex v_t is included in this choice as having degree 1. The graph of this process consists of a directed forest of $\Theta(\log t)$ components **whp** , where each component is rooted at a looped vertex. By grouping m consecutive vertices together we obtain $(G_m(t))_{t\geq 0}$, the process in which a new vertex and m new edges are added at each step. The scale-free model has many advantages, not least of which its simplicity of formulation. For fixed t, it can be reduced to a simple configuration model based on uar pairings. This allows exact calculation of probabilities using combinatorial arguments free from any considerations of the underlying process (see [2]). In [2] Bollobás, Riordan, Spencer and Tusnády prove that the degree sequence of scale-free graphs does indeed follow a power law distribution and give precise results for the proportional degree sequence. In [5] Bollobás and Riordan show that the diameter of the graph $G_m(t)$ constructed by the scale-free process is asymptotically $\log t/\log\log t$ **whp** . The work of Buckley and Osthus [7] generalizes the scale-free model to incorporate uar choices.

The web-graph model introduced by Cooper and Frieze in [8] generates a sequence of digraphs $(G(t))_{t\geq 0}$ where $G(0)$ is some (small) fixed digraph, and $G(t)$ is obtained from $G(t-1)$ by adding edges and vertices as follows. At step t, either a new vertex is added with edges joining it to $G(t-1)$ or extra edges are inserted within $G(t-1)$. If a new vertex v is added, each edge incident with v independently chooses its other endpoint vertex within $G(t-1)$ according to a two step experiment. At the first step of the experiment, the sampling method for choosing the endpoint is decided (uar or by preferential attachment). At the second step the endpoint vertex is sampled from $G(t-1)$. If extra edges are inserted within $G(t-1)$, each edge endpoint independently makes similar choices (precise details are given later). The number of edges added at any step is given by a distribution. This approach has the advantage that the model can be adjusted in network simulations to get a close fit to an observed power law.

Perhaps the most interesting general result of this model is that the power law parameter x can be written as $x = 1 + 1/\eta$ where η is the limiting ratio of the expected number of edge endpoints inserted by preferential attachment to the expected total degree. As an example, if a new vertex is added at each step with m new edges joined to the existing graph by preferential attachment, then after t steps the total degree is $2mt$ and mt edge endpoints have been inserted by preferential attachment. Thus $\eta = 1/2$ and $x = 3$, the standard model.

In the copy model of Kumar, Raghavan, Rajagopalan, Sivakumar, Tomkins and Upfal [12], [13] (sometimes called a duplication model) the new vertex, v, selects an existing vertex w uar. With some probability α an edge (v, w) is inserted, and with probability $1-\alpha$ the vertex v directs an edge from itself to a uar neighbour of the existing vertex w. This is equivalent to preferential attachment, in that the choice of neighbour vertex is proportional to its degree. This approach increases the local clustering of the vertices, a phenomenon observed in the WWW. In Section 5 we briefly remark on an alternative approach to increasing local clustering by inserting hypergraph edges.

In all these models the edges of the graph have an intrinsic orientation. In an undirected process the preferential attachment is based on total vertex degree. In a directed process, the in-degree and out-degree of vertices explicitly affect the generative procedure. The simplest example of this is the hub-authority process. Here the initial vertex of an inserted edge is chosen preferentially according to out-degree of existing vertices and the terminal vertex according to in-degree. This corresponds to the Hub-Authority model proposed by Kleinberg [11] for the WWW, in which hubs (vertices of large out-degree) point to authorities (vertices of large in-degree). Understanding the degree distribution of hub-authority web-graphs helps, perhaps, to explain the power law distributions of the in-degree and out-degree observed empirically for the WWW. For example [6] found parameters $x^- = 2.1$ and $x^+ = 2.7$ for the in-degree and out-degree respectively. We give results for the hub-authority model in Section 4 of this paper. These follow as a simple generalization of the proofs for the undirected case. Bollobás, Borgs, Chayes and Riordan [4] also consider a directed scale-free model of this type, generalized to include uar vertex choices.

In this paper we give results for two web-graph processes, the *undirected process* and the *hub-authority process*. We remark that it is straightforward to formulate and analyse more general directed models of preferential attachment based on a weighted mixture $ad^-(v) + bd^+(v)$ of in- and out degree at any vertex v. It is however difficult to summarize the results in a closed form and we do not consider these models further here.

The properties of web-graph processes we consider are the number of vertices of a given degree and the distribution of the degree $d(v,t)$ of a given vertex v at step t. The layout of this paper is as follows: In Section 2 we define the undirected process and state our results for this process. In Section 3 we prove these results for the undirected process. In Section 4 we define and give results for the hub-authority process. A brief summary of an undirected hypergraph process is given in Section 5.

Assertions made in this paper are in probability, and for sufficiently large t. Usually we qualify this explicitly with the abbreviation **whp** (with probability tending to 1 as $t \to \infty$). Throughout this paper we use ω to denote $\log t$. For presentation of asymptotic results we use $a(t) \sim b(t)$ to mean $\lim_{t\to\infty} a(t)/b(t) = 1$. In a different context, we also use $X \sim D$ to denote a random variable X with distribution D. A statement of the form (eg.) $f(x) = (1+O(y))g(x) + O(z)$ is usually treated as an inequality giving upper and lower bounds for $f(x)$ with error terms whose precision is a function of y, z. We also use such statements as shorthand notation, where each occurrence of $f(x)$ is to be replaced by $(1+O(y))g(x)+O(z)$ as appropriate. No precise functional relationship is implied in either case.

We use the following definitions and results relating to the Gamma function $\Gamma(x)$:
For real $x > 0$, $\Gamma(x) = \int_0^\infty w^{x-1}e^{-w}dw$ satisfies $\Gamma(x) = (x-1)\Gamma(x-1)$ and has asymptotic value $\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}(1+O(1/x))$. For real $a, b \geq 0$ the Beta Integral, $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. The generalized binomial coefficient is defined by

$$\binom{a+b}{a} = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\,\Gamma(b+1)}.$$

2 Undirected web-graph process.

Let $G(t) = (V(t), E(t))$ denote the graph of the process at the end of step t. We label vertices according to the step at which they are added to the graph. Unless a vertex is added at each step we will have $|V(t)| < t$. At least one edge is always added at each step, so that $|E(t)| \geq t$.

2.1 Definition of the process

There is an initial connected (di)-graph $G(0)$ of finite size. Loops and multiple edges are allowed in $G(0)$ and can be formed at any step. At each step $t = 1, 2, ...$, an independent choice is made between a NEW and an OLD procedure, according to a Bernoulii random variable. Let α be the probability the NEW procedure is followed and $\beta = 1 - \alpha$ the probability the OLD procedure is followed. We assume $\alpha > 0$ so that the expected size of the vertex set increases with t.

NEW **procedure.** Chosen with probability α at step t. A new vertex v_t is added with $m(t)$ edges directed to $G(t-1)$. For convenience we refer to vertex v_t by its step label t (although not every step will necessarily have a corresponding vertex). Let p_m denote the probability $\mathbf{Pr}(m(t) = m)$ that m edges are added to v_t at step t. Thus the number $m(t)$ of edges originating from v_t is a random variable. We assume $m(t) \geq 1$, and that $m(t)$ is sampled from a *finite* distribution $\boldsymbol{p} = (p_1, ..., p_m, ..., p_L)$, where $L < \infty$ and $p_m \geq 0$ constant. Let $\overline{m} = \mathbf{E}\, m(t)$. We have that $\overline{m} \geq 1$. The degree $m(t)$ of vertex v_t when added is denoted by $d(t, t)$. Similarly $d(v, t)$ is the degree of vertex v at step t.

Each new edge $e_i, i = 1, ..., m$ makes an independent decision to select its terminal vertex w in $G(t-1)$ either by preferential attachment or uar with probabilities $A_1,\ A_2 = 1 - A_1$ respectively. Thus

$$\mathbf{Pr}(w \text{ is selected by } e_i) = p_A(w,t) = A_1 \frac{d(w, t-1)}{2|E(t-1)|} + A_2 \frac{1}{|V(t-1)|}, \tag{1}$$

where $d(w, t-1)$ is the degree of w at the end of step $t-1$.

OLD **procedure.** Chosen with probability $\beta = 1 - \alpha$ at step t. Insert $M(t)$ edges into $G(t-1)$ according to a finite distribution $\boldsymbol{q} = (q_1, ..., q_M, ..., q_L)$. Let $\overline{M} = \mathbf{E}\, M(t)$. The choice of initial vertex (resp. terminal vertex) of each inserted edge is made independently with probability p_B (resp. p_C) from uar/preferential rules similar to (1), but with parameters B_1, B_2 (resp C_1, C_2).

Thus in all cases $Z = A, B, C$ we have

$$p_Z(v,t) = Z_1 \frac{d(v, t-1)}{2|E(t-1)|} + Z_2 \frac{1}{|V(t-1)|}. \tag{2}$$

The model differs from that described in [8] where edges inserted by the OLD procedure were required to have a single initial vertex.

2.2 The undirected process: Statement of results

In this section we state and discuss results for the expected number of vertices of a given degree (Theorem 1) and for the distribution of degree of a given vertex (Theorem 2). These theorems follow from proofs given in Section 3, as is explained at the end of the current section.

The degree distribution of the undirected process depends on two parameters η, ν given by

$$\eta = \frac{\alpha \overline{m} A_1 + \beta \overline{M}(B_1 + C_1)}{2(\alpha \overline{m} + \beta \overline{M})} \tag{3}$$

$$\nu = \frac{\alpha \overline{m} A_2 + \beta \overline{M}(B_2 + C_2)}{\alpha}. \tag{4}$$

The parameters η, ν summarize the preferential attachment and uar aspects of the process respectively. The parameter η is the limiting ratio of the expected number of edge endpoints inserted by preferential attachment to the expected total degree of the graph. Similarly, ν is the average number of edges inserted uar per vertex. We allow $\nu = 0$ but we always assume $0 < \eta < 1$. Requiring $\alpha > 0$ implies that $\eta < 1$. As $\overline{m}, \overline{M} \geq 1$, requiring $A_1 + B_1 + C_1 > 0$ means that $\eta > 0$.

For the undirected process the power law parameter is $x = 1 + 1/\eta$. It is impossible to generate processes with parameter $\eta > \frac{1}{2}$ $(2 < x < 3)$, based solely on the addition of new vertices at each step. Rather, it requires edges to be inserted between existing vertices. Also, it is impossible to generate a process with $\eta < \frac{1}{2}$ $(x > 3)$ based solely on preferential attachment, rather it requires some uar choice of vertices. If only new vertices with m edges are added preferentially $(\alpha = 1, p_m = 1, A_1 = 1)$ then $\eta = 1/2, \nu = 0$ and $x = 3$. This corresponds to the scale-free model of Bollobás and Riordan [5]. The model of Buckley and Osthus, [7] generalizes the scale-free model to incorporate uar choices. It is equivalent to an undirected process with $\eta = 1/(2(a+1)), \nu = ma/(a+1)$, where a the *initial attractiveness* [7] is a natural number. In other words, the preferential choice is made by an edge with probability $A_1 = 1/(a+1)$ and the uar choice with probability $A_2 = a/(a+1)$ (put $\alpha = 1, \overline{m} = m$ in (3), (4)).

As previous remarked, $m(v) = d(v, v)$ the degree of vertex v when first added at step v, is a random variable. We define a vertex dependent parameter

$$\xi(v) = m(v) + \nu/\eta,$$

and write $\xi(m)$ or simply ξ for $\xi(m(v))$ when the context is clear.

We now consider the proportion of vertices of a given degree. Let

$$n_m(l) = \frac{\Gamma(\xi + 1/\eta)}{\eta \Gamma(\xi)} \frac{\Gamma(l + \xi)}{\Gamma(l + \xi + 1 + 1/\eta)}. \tag{5}$$

For $l \geq 0$ define $D_t(l, m)$ as the number of vertices of $V(t)$ with $d(v, v) = m$ and $d(v, t) = l + m$. Recall that p_m is the probability that $d(v, v) = m$ and that $\omega = \log t$.

Theorem 1. *For $\eta < 1/2$ let $l^\star = t^{\eta/3}$, and for $\eta \geq 1/2$ let $l^\star = t^{1/6}/\omega^2$. For $0 \leq l \leq l^\star$ we have:*

(a) Expected degree sequence

$$\mathbf{E}\ D_t(l,m) = \alpha p_m n_m(l) t\ \left(1 + O\left(\tfrac{1}{\log t}\right)\right).$$

(b) Concentration

$$\mathbf{Pr}\left(|D_t(l,m) - \mathbf{E}\ D_t(l,m)| \geq \frac{\mathbf{E}\ D_t(l,m))}{\sqrt{\log t}}\right) = O\left(\frac{1}{\log t}\right).$$

As an example of the results of this theorem, consider a process in which a new vertex with m edges is added preferentially at each step ($\alpha = 1, p_m = 1, A_1 = 1$). Thus $\eta = 1/2, \nu = 0, \xi = m$ and $\mathbf{E}\ D(l,m)/t \sim 2m(m+1)/((l+m+2)(l+m+1)(l+m))$ a result obtained in [2] for scale-free graphs. Define $N(l,m)$, the proportion of vertices of initial degree m, and added degree l by $N(l,m) = \mathbf{E}\ D(l,m)/\mathbf{E}\ |V(t)|$. It follows from Section 3.1 that $\mathbf{E}\ |V(t)| = \alpha t(1 + o(1))$. Thus $N(l,m) \sim p_m n_m(l)$. In general, a more convenient expression for $n_m(l)$ is

$$n_m(l;\eta,\nu) = \frac{((l+m-1)\eta + \nu)\cdots(m\eta + \nu)}{((l+m)\eta + \nu + 1)\cdots(m\eta + \nu + 1)}. \tag{6}$$

For a process where $p_m = 1$, the expected proportion of vertices of degree $l + m, l \geq 0$ tends to this quantity. To get the asymptotics, we note from (5) that

$$n_m(l) = \left(1 + O\left(\tfrac{1}{l}\right)\right) \tfrac{\Gamma(\xi+1/\eta)}{\eta\Gamma(\xi)}\ l^{-\left(1+\frac{1}{\eta}\right)}, \tag{7}$$

and as $l \to \infty$, $N(l,m) \sim Cl^{-(1+1/\eta)}$, a power law with parameter $1 + 1/\eta$. Details of the process are hidden in the model specific constant, $C(\eta,\nu,m) = p_m\Gamma(m + \frac{\nu+1}{\eta})/(\eta\Gamma(m + \frac{\nu}{\eta}))$.

We note that $\sum_l n_m(l) = 1$. Indeed, using $(1/\eta)\Gamma(1/\eta) = \Gamma(1 + 1/\eta)$ we see that $n_m(l)$ can be written as

$$n_m(l) = \frac{B(l+\xi, 1+1/\eta)}{B(\xi, 1/\eta)}.$$

However $\sum_{l\geq 0} B(l+\xi, 1+1/\eta) = B(\xi, 1/\eta)$ giving $\sum_l n_m(l) = 1$.

If the total probability $(A_1 + B_1 + C_1)$ of preferential attachment in the process tends to zero, then the degree sequence of the web-graph process should approach that of the equivalent graph process in which vertices are chosen uar. Taking the limit of (6) as $\eta \to 0$ we obtain

$$\lim_{\eta\to 0} n_m(l;\eta) \sim \frac{1}{\nu+1}\left(\frac{\nu}{\nu+1}\right)^l,$$

a geometric sequence typical of a uar graph process (see also [2]).

We now turn our attention to the distribution of degree $d(v,t)$ of a given vertex v. We remark that the expected degree of vertex v is $\Theta((t/v)^\eta)$. This follows from (16) and (17) of Section 3 which give

$$\mathbf{E}\ (d(v,t+1) \mid d(v,t)) \sim d(v,t) + \frac{\eta d(v,t)+\nu}{t}.$$

When $d(v,v)=m$ this has solution

$$\mathbf{E}\ d(v,t) \sim m + \left(m+\tfrac{\nu}{\eta}\right)\left(\left(\frac{t}{v}\right)^{\eta} - 1\right).$$

Over a wide range, the distribution of the degree of vertex v is effectively negative binomial, which is the main result of the following theorem.

Theorem 2. *Let $l^*(v) = \min\{t^\eta/\omega^4,\ v^{1/2}/\omega^4,\ t^{1/2}(v/t)^\eta/\omega^4\}$, and let K be a positive constant.*

(a) For $\omega^8 \le v \le (t - t/\omega)$ and $0 \le l \le l^*(v)$ or for $(t-t/\omega) < v < t$ and $l = 0, 1$,

$$\begin{aligned}&\mathbf{Pr}(d(v,t) = m+l \mid d(v,v)=m)\\ &\quad = \left(1+O(\tfrac{1}{\omega^2})\right)\binom{l+\xi(m)-1}{l}\left(\frac{v}{t}\right)^{\eta\xi(m)}\left(1-\left(\frac{v}{t}\right)^{\eta}\left(1+O(\tfrac{1}{\omega^2})\right)\right)^{l} + O(v^{-K}).\end{aligned}$$

(b) For $(t-t/\omega) < v < t$ and $l \ge 2$,

$$\mathbf{Pr}(d(v,t) = m+l \mid d(v,v)=m) = O(l^{\xi(m)-1}/\omega^l) + O(t^{-K}).$$

(c) For v such that $l^*(v) \ge \omega^2(t/v)^\eta$,

$$\mathbf{Pr}(\exists t \ge v : d(v,t) \ge K\omega(t/v)^\eta) = O(v^{-K+2}).$$

Theorem 2 says that for values of v satisfying (c), the almost sure maximum degree of vertex v is at most $(t/v)^\eta K\omega$. In this case, for $v \ge \omega^8$, (a) and (b) characterize the distribution of $d(v,t)$. As an example, choosing $(\alpha=1, p_m=1, A_1=1)$ gives $\eta = 1/2, \nu = 0$ and power law parameter $x=3$, which corresponds to the models studied in eg. [1], [2]. Thus $l^*(v) = v^{1/2}/\omega^4$ and for $v \ge t^{1/2}\omega^6$, (c) is satisfied. For such v, the above theorem gives a full description of the degree distribution.

We now explain how the proofs of Theorems 1,2 will follow from the results of Section 3. The size $|V(t)|$ of the vertex set (resp. $|E(t)|$ of the edge set) at step t is the sum of independent random variables giving the number of vertices (resp. edges) added at each step. The concentration of these variables is given by (8), (10). For processes where $|V(\tau)|, |E(\tau)|$ are close to their expected values for $v \le \tau \le t$, (*v-good* processes), Theorem 3 establishes the distribution of the degree of vertex v for values $d(v,t) = m+l$ where $l \le l^*(v)$. Theorem

2(a),(b) will follow from Theorem 3 and the Hoeffding bound of $O(v^{-K'})$ for the probability that a process is not v-good at some step $t \geq v$. Theorem 2(c) follows from Lemma 1 which is given at the end of Section 3.2. Theorem 1 follows directly from Lemma 2 of Section 3.3. This lemma estimates the expected value of $D_t(l,m)$ from Theorem 3(a)(i) by fixing l, m and summing $\mathbf{Pr}(d(v,t) = m + l)$ over $v \leq t$. Theorems 1,2 exist mainly to summarize results. The content of the paper is in the proof of Theorem 3. Finally we remark that the notation $l^\star$ used for upper bounds on degree in Theorem 1 is to be seen as distinct from the vertex dependent upper bound $l^*(v)$ is used in Theorem 2.

3 The undirected process: Proof of Theorems 1, 2

3.1 Good processes

Let $\boldsymbol{G}(t) = \{G(t) = (V(t), E(t))\}$ be the set of graphs obtained at the end of step t from processes $(G(\tau), \tau \leq t)$. The values of $|V(t)|$, $|E(t)|$ (the total number of vertices and edges) are the sum of independent bounded random variables. At each step, with probability $\alpha > 0$ a new vertex is added and at least one edge is always added. The vertex set $V(t)$ can be partitioned into sets $V_m(t)$ of vertices of initial degree m (ie. $d(v,v) = m$). Thus we have

$$\begin{aligned}\mu_V &= \mathbf{E}\,|V(t)| = t\alpha + |V(0)| \\ \mu_{V,m} &= \mathbf{E}\,|V_m(t)| = t\alpha p_m + |V_m(0)| \\ \mu_E &= \mathbf{E}\,|E(t)| = t(\alpha\overline{m} + (1-\alpha)\overline{M}) + |E(0)|.\end{aligned}$$

Choosing $K > 0$ and $\epsilon = \sqrt{K(1 + \log t)/t}$, by the Hoeffding Inequality (eg. page 162 of [14])

$$\mathbf{Pr}\left(||V(t)| - \mu_V| \geq \epsilon\mu_V\right) = O\left(t^{-2K\alpha^2}\right) \tag{8}$$

$$\mathbf{Pr}\left(||V_m(t)| - \mu_{V,m}| \geq \epsilon\mu_{V,m}\right) = O\left(t^{-2K(\alpha p_m)^2}\right) \tag{9}$$

$$\mathbf{Pr}\left(||E(t)| - \mu_E| \geq \epsilon\mu_E\right) = O\left(t^{-2K/L^2}\right). \tag{10}$$

Inequalities (8), (10) hold simultaneously for all $t \geq v$ with probability $1 - O(v^{-K'})$, where $K' = 2K\min(\alpha^2, L^{-2}) - 1$. We assume that K has been chosen sufficiently large, so that $K' \geq 4$.

Referring to the definition of the process in Section 2.1, let $1_\alpha(t) = 1$ if procedure NEW occurs at step t and $1_\alpha(t) = 0$ if procedure OLD occurs. Let $c_t = (1_\alpha(t), m(t), M(t))$ give the choice of procedure and number of edges inserted at step t. Let $\boldsymbol{c}(t) = (c_1, ..., c_t)$ be the history of these choices and let $\boldsymbol{C}(v,t)$ be the set of *v-good sequences* $\boldsymbol{c}(t)$ with $(|V(\tau)|, |E(\tau)|)$ satisfying

(8), (10) for $v \le \tau \le t$. Assume a new vertex labeled v, of initial degree $m(v) = m$ was added at step v. Thus $1_\alpha(v) = 1$ and $c_v = (1, m, 0)$. Let $I(t) = I_{m,v}(t)$ be given by

$$I_{m,v}(t) = \{\boldsymbol{c}(t) \in \boldsymbol{C}(v,t), 1_\alpha(v) = 1,\ d(v,v) = m\}. \tag{11}$$

The degree d of vertex v evolves as a Markov process with states $(d, |V|, |E|)$. The step t and what happens in the rest of the graph is irrelevant. However, because we are dealing with graph processes $(G(t))_{t\ge0}$ we distinguish the step t and by the notation $I(t)$, those processes which have always been good from step v onward, giving a state description $(t, d(t), |V(t)|, |E(t)|, S(t))$ where $S = I, \overline{I}$. For good processes we can approximate $|V(t)|,\ |E(t)|$ by their expected values, incurring a multiplicative error of $1 + O(\sqrt{\log t/t})$ in doing so. By calculating the transition probabilities $\mathbf{Pr}(t+1, d(t+1), I(t+1) \mid t, d(t), I(t))$ and adding over all paths starting in $(v, m, I(v))$ and ending in $(t, d(t), I(t))$ we can estimate $\mathbf{Pr}(t, d(t), I(t) \mid v, m, I(v))$ which is what we need.

3.2 Proof of Theorem 2: Distribution of vertex degree

To simplify notation let

$$\mathbb{P}(d(v,t) = m + l) = \mathbf{Pr}(d(v,t) = m + l, I(t) \mid d(v,v) = m, I(v)),$$

and assuming $v < w$, let

$$\mathbb{P}(d(v,t) = m + l,\, d(w,t) = m' + l') = \\ \mathbf{Pr}(d(v,t) =\ m + l,\ d(w,t) = m' + l', I(t) \mid d(v,v) = m,\ d(w,w) = m', I(v)).$$

For brevity of notation we define

$$\pi_{l,m}(v,t;\eta,\nu) = \left(1 + O\left(\tfrac{1}{\omega^2}\right)\right) \binom{l+\xi-1}{l} \left(\frac{v}{t}\right)^{\eta\xi} \left(1 - \left(\frac{v}{t}\right)^{\eta} \left(1 + O\left(\tfrac{1}{\omega^2}\right)\right)\right)^l. \tag{12}$$

Recalling the discussions in the introduction, we remark that $\pi_{l,m}(v,t;\eta,\nu)$ is not a function of (v,t,l,m) but rather a shorthand notation for the expression on the right hand side. The constants in the $O(.)$ terms may vary, even for given v,t,l,m in subsequent uses of the expression although they are always bounded in absolute value (see eg. parts (a)(i) and (b) of the theorem below). Finally we remark that eg. in Theorem 3 (a) (i) the statement $\mathbb{P}(d(v,t) = m + l) = \pi_{l,m}(v,t)$ is to be read as an inequality with bounds above and below given by (12) for suitable choices of the $O(1/\omega^2)$ terms.

Theorem 3. *Let* $l^*(v) = \min\{t^\eta/\omega^4, v^{1/2}/\omega^4, t^{1/2}(v/t)^\eta/\omega^4\}$.

(a) (i) For $\omega^8 \le v \le (t - t/\omega)$ and $0 \le l \le l^*(v)$ or for $(t - t/\omega) < v < t$ and $l = 0, 1$,

$$\mathbb{P}(d(v,t) = m + l) = \pi_{l,m}(v,t) + O(v^{-K'}). \tag{13}$$

(a) (ii) For $(t - t/\omega) < v < t$ and $l \ge 2$,

$$\mathbb{P}(d(v,t) = m + l) = O(\gamma^l)\pi_{l,m}(v,t) + O(v^{-K'}), \tag{14}$$

where γ is a constant of the process independent of v, t, l.

(b) For $\omega^8 \le v < w < (t - t/\omega)$ and $l \le l^*(v), l' \le l^*(w)$

$$\mathbb{P}(d(v,t) = m + l, d(w,t) = m' + l') = \pi_{l,m}(v,t)\pi_{l',m'}(w,t) + O(v^{-K'}). \tag{15}$$

Proof

Let $X(t)$ count the number of edges selecting vertex v at step t. The distribution of $X(t)$ given $(d(v,t-1), |V(t-1)|, |E(t-1)|)$ is

$$X(t) \sim 1_\alpha(t)\ \mathrm{Bin}(m(t), p_A(t))\ +\ 1_\beta(t)\ [\mathrm{Bin}(M(t), p_B(t)) + \mathrm{Bin}(M(t), p_C(t))],$$

where the parameters are described in Section 2.1, and $\mathrm{Bin}(m,p)$ is the Binomial distribution. Conditional on $d(v,t) = m + j$ and on $|V(t)|, |E(t)|$ satisfying (8) and (10), recalling the definitions (1-4) we have

$$\begin{aligned}\mathbf{E}\ (X(t+1) \mid d(v,t) = m + j) &= \alpha\ p_A \mathbf{E}_{\boldsymbol{p}} m(t+1) + \beta\ (p_B + p_C)\mathbf{E}_{\boldsymbol{q}} M(t+1) \qquad (16)\\ &= \frac{\eta(m+j)+\nu}{t}\left(1 + O\left(\sqrt{\tfrac{\log t}{t}}\right)\right), \qquad (17)\end{aligned}$$

where $\boldsymbol{p}$ (resp. $\boldsymbol{q}$) is the distribution of $m(t+1)$ (resp. $M(t+1)$) the number of edges added by the NEW (resp. OLD) procedure at step $t+1$. Also

$$\begin{aligned}\mathbf{Pr}(X(t+1) =&k \mid d(v,t) = m + j) \qquad (18)\\ =&\alpha\ \mathbf{E}_{\boldsymbol{p}}\binom{m(t+1)}{k} p_A^k (1-p_A)^{m(t+1)-k} \qquad (19)\\ &+\beta\ \mathbf{E}_{\boldsymbol{q}}\sum_i \binom{M(t+1)}{i}\binom{M(t+1)}{k-i} p_B^i p_C^{k-i}(1-p_B)^{M-i}(1-p_C)^{M-(k-i)}. \qquad (20)\end{aligned}$$

In particular

$$\begin{aligned}\mathbf{Pr}(X(t+1) = 0 \mid d(v,t) = m + j) &= 1 - \frac{\eta(m+j)+\nu}{t}\left(1 + O(\epsilon(t,j))\right) \qquad (21)\\ \mathbf{Pr}(X(t+1) = 1 \mid d(v,t) = m + j) &= \frac{\eta(m+j)+\nu}{t}\left(1 + O(\epsilon(t,j))\right), \qquad (22)\end{aligned}$$

where

$$\epsilon(\tau, j) = \sqrt{\log \tau/\tau} + (m+j)/\tau. \tag{23}$$

Let $\rho = \max(p_A, p_B, p_C)$. The number of edges k added at any step is finite ($m(t), M(t) \leq L$) and $\eta > 0$ as $A_1 + B_1 + C_1 > 0$. From (18)-(20), $\mathbf{Pr}(X(t+1) = k) = O(\rho^k)$, and $\rho = O((m+j)/t)$ and so $\mathbf{Pr}(X(t+1) = k) = O(\eta(m+j)/t)^k)$. Thus there exists a constant $\gamma > 0$ depending only on the process, such that for any $2 \leq k \leq L$

$$\mathbf{Pr}(X(t+1) = k \mid d(v,t) = m+j) \leq \frac{\gamma}{t^k}(\eta(m+j)+\nu)\cdots(\eta(m+j+k-1)+\nu). \tag{24}$$

Remark. As we condition on membership of $I(t)$ at each step, we assume the continued goodness $I(t+1)$ and for $k \geq 0$ we use transition probabilities

$$\mathbf{Pr}(d(v,t+1) = m+j+k,\ I(t+1) \mid d(v,t) = m+j, I(t)), \tag{25}$$

in the calculations below. The probabilities derived above for the event $X(t+1) = k$, do not exclude the possibility that a sequence $\boldsymbol{c}(t)$ which is good at step t, goes bad at step $t+1$ (ie. $\boldsymbol{c}(t+1) \in \overline{I}(t+1)$). This means expressions (21), (22) etc. overestimate, $\mathbf{Pr}(X(t+1) = k, I(t+1) \mid m+j, I(t))$.

To deal with this, we note the following inequality: Let $p_i + q_i \leq 1,\ i = 1, ..., s$. Let $f = p_1 p_2 \cdots p_s$, $g = (p_1+q_2)(p_2+q_2)\cdots(p_s+q_s)$ and $h = q_1 + p_1 q_2 + \cdots + p_1 p_2 \cdots p_{s-1} q_s$. Then $f \leq g \leq f + h$. In the context here, $(p_s + q_s)$ is the left hand side of (21), (22), (24), with $t+1 = s+v$ and

$$p_s = \mathbf{Pr}(X(s+v) = k_s, I(s+v) \mid d(v, s+v-1), I(s+v-1)).$$

The value of $d(v,t+1)$ is completely determined by $k_1, ..., k_s$ the number of edges it acquires at steps $v+1, ..., t+1$, and

$$\sum_{(k_1,...,k_s)} h(k_1,...,k_s) \leq \mathbf{Pr}(\exists \tau \geq v \text{ sequence } \boldsymbol{c}(\tau) \text{ goes bad at } \tau) = O(v^{-K'}).$$

By adding $O(v^{-K'})$ to the estimates derived below using (g instead of f), we can correct the error introduced from expressions (21), (22), (24). Finally we note that for brevity we often drop the explicit mention of the $I(t)$ terms in what follows.

For fixed v, t, suppose $d(v,t) = m+l$ and let $\mathbf{T} = (\mathbf{T}_j,\ j = 1, ..., l)$ give the steps (if any) at which the degree of v changed. Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_l)$ denote a particular value of $\mathbf{T}$, so that τ_j is the step at which $d(v, \tau_j)$ changed from $m+j-1$ to $m+j$ in this case. If more than one edge hit v during step τ_j, the value τ_j is repeated the appropriate number of times in $\boldsymbol{\tau}$. We define $\tau_0 = v$ and $\tau_{l+1} = t$. For $v < \tau \leq t$ let

$$\begin{aligned} J_1 =& \{\boldsymbol{\tau} : \tau_1 < \tau_2 < \cdots < \tau_l\} \\ J_2 =& \{\boldsymbol{\tau} : \tau_1 \leq \tau_2 \leq \cdots \leq \tau_l, \text{ and at least one } \tau_i = \tau_{i+1}\}, \end{aligned} \tag{26}$$

be the sequences of possible transitions. Thus

$$\mathbf{Pr}(d(v,t)=m+l \mid d(v,v)=m) = \sum_{\boldsymbol{\tau} \in J_1 \cup J_2} \mathbf{Pr}(\mathbf{T}=\boldsymbol{\tau}).$$

Using $\mathbb{P}(m+l)$ to denote $\mathbf{Pr}(d(v,t)=m+l, I(t) \mid d(v,v)=m, I(v))$ we have

$$\mathbb{P}(m+l) = \mathbb{P}(m+l, J_1) + \mathbb{P}(m+l, J_2).$$

Let $\Psi_j = \mathbf{Pr}(X(T)=0,\ \tau_j < T < \tau_{j+1},\ I(\tau_{j+1}-1) \mid I(\tau_j))$ (with $\tau_l < T \le \tau_{l+1} = t$ when $j=l$). If $\tau_{j+1} = \tau_j$ or $\tau_j + 1$ let $\Psi_j = 1$. If $\tau_{j+1} \ge \tau_j + 2$, then from (21)

$$\Psi_j = \prod_{\tau_j < T < \tau_{j+1}} \left(1 - \frac{\eta(m+j)+\nu}{T} \left(1+O(\epsilon(T,j))\right)\right). \tag{27}$$

Let

$$\delta(\tau, j) = (m+j)\sqrt{\log \tau / \tau} + (m+j)^2/\tau, \tag{28}$$

then

$$\begin{aligned} \Psi_j &= \exp\left(\left(-(\eta(m+j)+\nu) \sum_{\tau_j < T < \tau_{j+1}} \frac{1}{T}\right) + O(\delta(\tau_j, j))\right) \\ &= \left(\frac{\tau_j}{\tau_{j+1}}\right)^{\eta(m+j)+\nu} (1 + O(\delta(\tau_j, j))). \end{aligned} \tag{29}$$

By requiring $l \le l^*(v) \le \sqrt{v}/\omega^4$ we have, for all $v \le \tau \le t$ and $0 \le j \le l$ that

$$\delta(\tau, j) = o\left(\frac{1}{\omega^2}\right). \tag{30}$$

Estimating $\mathbb{P}(d(v,t) = m+l, J_1)$.
For $0 \le j \le l-1$, let $\Phi_j(t+1) = \mathbf{Pr}(X(t+1)=1,\ I(t+1) \mid d(v,t)=m+j, I(t))$. Thus from (22)

$$\Phi_j(t+1) = \frac{\eta(m+j)+\nu}{t} \left(1+O(\epsilon(t,j))\right).$$

Let $\Phi_j = \Phi(\tau_{j+1})$, and let $\Phi_l = 1$. Let $F(\boldsymbol{\tau})$ denote $\mathbf{Pr}(d(v,t)=m+l$ and $\boldsymbol{\tau},\ I(t) \mid d(v,v) = m, I(v))$. Let $\mathbf{Pr}(\boldsymbol{T}_j = \tau_j \mid \boldsymbol{T}_{j-1} = \tau_{j-1})$ be the probability that the transition to $m+j$ occurs at τ_j given the transition to $m+j-1$ at τ_{j-1} (and the goodness continues). Thus

$$F(\boldsymbol{\tau}) = \Psi_l \prod_{j=1}^{l} \mathbf{Pr}(\boldsymbol{T}_j = \tau_j \mid \boldsymbol{T}_{j-1} = \tau_{j-1}) = \prod_{j=0}^{l} \Psi_j\ \Phi_j. \tag{31}$$

Ignoring for the moment the multiplicative error terms, we see that $F(\boldsymbol{\tau})$ is given by

$$\left(\frac{v}{\tau_1}\right)^{\eta m+\nu} \frac{\eta m+\nu}{\tau_1} \left(\frac{\tau_1}{\tau_2}\right)^{\eta(m+1)+\nu} \frac{\eta(m+1)+\nu}{\tau_2} \cdots \left(\frac{\tau_{l-1}}{\tau_l}\right)^{\eta(m+l-1)+\nu} \frac{\eta(m+l-1)+\nu}{\tau_l} \left(\frac{\tau_l}{t}\right)^{\eta(m+l)+\nu}. \tag{32}$$

Recall that $\xi = m + \nu/\eta$. We cancel repeated values of τ_j to give

$$F(\boldsymbol{\tau}) = (1 + O(\delta(t, l)))\, \frac{\Gamma(l + \xi)}{\Gamma(\xi)} \left(\frac{v}{t}\right)^{\eta\xi} \prod_{j=1}^{l} \frac{\eta\, \tau_j^{\eta-1}}{t^\eta} \left(1 + O(\delta(\tau_j, j))\right). \tag{33}$$

Thus

$$\mathbb{P}(m + l, J_1) = (1 + O(\delta(t, l)))\, \frac{\Gamma(l + \xi)}{\Gamma(\xi)} \left(\frac{v}{t}\right)^{\eta\xi} P_1,$$

where

$$P_1 = \sum_{\boldsymbol{\tau}\in J_1} \prod_{j=1}^{l} \frac{\eta\, \tau_j^{\eta-1}}{t^\eta} \left(1 + O(\delta(\tau_j, j))\right). \tag{34}$$

For $b_j \ge 0$

$$(b_v + \cdots + b_t)^k - ({b_v}^2 + \cdots + {b_t}^2)\binom{k}{2}(b_v + \cdots + b_t)^{k-2} \le k! \sum_{i_1<\cdots<i_k} b_{i_1} \cdots b_{i_k} \le (b_v + \cdots + b_t)^k.$$

We replace the term $\delta(\tau_j, j)$ in $F(\boldsymbol{\tau})$ with $\delta(\tau_j, l)$ and choose

$$b_\tau = \frac{\eta\tau^{\eta-1}}{t^\eta} \left(1 + O(\delta(\tau, l))\right). \tag{35}$$

Thus

$$P_1 = \frac{1}{l!} \left((b_v + \cdots + b_t)^l + O(l^2)({b_v}^2 + \cdots + {b_t}^2)(b_v + \cdots + b_t)^{l-2}\right). \tag{36}$$

For $\eta \ne 1/2$, using (28) and the definition of $l^*(v)$,

$$\begin{aligned} b_v + \cdots + b_t &= \sum_{v\le\tau\le t} \frac{\eta\tau^{\eta-1}}{t^\eta}(1 + O(\delta(\tau, l))) \\ &= 1 - \left(\frac{v}{t}\right)^\eta + O\left(\frac{l\sqrt{\log t}}{\sqrt{t}} + \frac{l^2}{t} + \left(\frac{v}{t}\right)^\eta \left(\frac{l\sqrt{\log v}}{\sqrt{v}} + \frac{l^2}{v}\right)\right) \\ &= 1 - \left(\frac{v}{t}\right)^\eta \left(1 + O\left(\tfrac{1}{\omega^2}\right)\right). \end{aligned}$$

For $\eta = 1/2$ the error term in the second line is $O(l(\log t)^{3/2} + l^2/\sqrt{v})/\sqrt{t}$, and the final line still holds.

For $1 \le v \le t(1 - 1/\omega)$ and $l \le \min\{t^\eta/\omega^4, t^{1/2}/\omega^4\}$ we prove below that

$$\sum_{\tau=v}^{t} b_\tau^2 = o\left(\frac{1}{\omega^2 l^2}\right) \left(\sum_{\tau=v}^{t} b_\tau\right)^2. \tag{37}$$

Thus, we obtain

$$P_1 = \left(1+O\left(\tfrac{1}{\omega^2}\right)\right)\frac{1}{l!}\left(1-\left(\frac{v}{t}\right)^{\eta}\left(1+O\left(\tfrac{1}{\omega^2}\right)\right)\right)^{l}, \tag{38}$$

completing the proof that $\mathbb{P}(m+l,J_1)=\pi_{l,m}(v,t)$ (see (12)).

Returning to the proof of (37), let

$$g(v,t)=\begin{cases}\frac{\eta^2}{1-2\eta}\left(\frac{1}{v}\left(\frac{v}{t}\right)^{2\eta}-\frac{1}{t}\right) & \eta<1/2\\ \frac{1}{4t}\log(t/v) & \eta=1/2\\ \frac{\eta^2}{2\eta-1}\left(\frac{1}{t}-\frac{1}{v}\left(\frac{v}{t}\right)^{2\eta}\right) & \eta>1/2\end{cases}$$

Using (30) we have that

$$b_v{}^2+\cdots+b_t{}^2 = g(v,t)\left(1+O\left(\tfrac{1}{\omega^2}\right)\right). \tag{39}$$

As $v\le t(1-1/\omega)$ then $\sum b\ge\Theta(1/\omega)$ and the inequality (37) is implied by $l^2g(v,t)=o(\omega^{-4})$. In all cases $g(v,t)$ is monotone decreasing in v. For $\eta<1/2$, $g(v)=O(t^{-2\eta})$ and we require $l=o(t^{\eta}/\omega^2)$ which we have assumed. For $\eta=1/2$, $g(v)=O(t^{-1}\log t)$ and we use $l=o(t^{1/2}/\omega^3)$. For $\eta>1/2$, $g(v)=O(t^{-1})$.

Estimating $\mathbb{P}(d(v,t)=m+l,J_2)$.
Assume $1\le v\le t(1-1/\omega)$. The derivation of the expression for the upper bound on $\sum_{\boldsymbol{\tau}\in J_2}F(\boldsymbol{\tau})$ is similar to the J_1 case except that some values τ_j are repeated at least twice. As before, (see (33-34)) we will separate off the fixed terms by writing

$$\mathbb{P}(m+l,J_2)=\sum_{\boldsymbol{\tau}\in J_2}F(\boldsymbol{\tau})=(1+O(\delta(t,l)))\frac{\Gamma(l+\xi)}{\Gamma(\xi)}\left(\frac{v}{t}\right)^{\eta\xi}\ P_2.$$

We now obtain an upper bound for P_2. Let $\tau=\tau_j$ be repeated $r=r_j$ times, $(r\ge 2)$. From (24), (29) and (32) the equivalent entry in P_2 is bounded above by

$$\frac{\gamma}{\tau^r}\left(\eta\frac{\tau^{\eta}}{t^{\eta}}\right)^r(1+O(\delta(\tau,l))),$$

which can be written as $\gamma(b_\tau)^r$ where b_τ is from (35). Let $r_j=2k_j+m_j$, $(m_j=0,1)$ and let $k=\sum_j k_j$, then from (37),

$$\begin{aligned}P_2 &\le \sum_{k=1}^{\lceil l/2\rceil}\gamma^k(b_v{}^2+\cdots+b_t{}^2)^k\frac{(b_v+\cdots+b_t)^{l-2k}}{(l-2k)!}\\ &\le \frac{(b_v+\cdots+b_t)^l}{l!}\sum_{k\ge1}\left(\frac{\gamma l^2(b_v{}^2+\cdots+b_t{}^2)}{(b_v+\cdots+b_t)^2}\right)^k\\ &= o\left(\frac{1}{\omega^2}\right)P_1,\end{aligned}$$

and hence $\mathbb{P}(m+l, J_2) = \mathbb{P}(m+l, J_1)o(1/\omega^2)$.

We now complete the derivation of part (a) of Theorem 3.

Case $1 \le v \le t(1-1/\omega)$**.**

$$\mathbb{P}(m+l, J_1) + \mathbb{P}(m+l, J_2) = \mathbb{P}(m+l, J_1)(1+o(1/\omega^2)) = \pi_{l,m}(v,t). \tag{40}$$

Case $t(1-1/\omega) < v < t$ **and** $l = 0, 1$**.** Here $J_2 = \emptyset$, $P_2 = 0$ and $\binom{l}{2} = 0$. The result is immediate from the (simplified) estimate for $\mathbb{P}(m+l, J_1)$.

Case $t(1-1/\omega) < v < t$ **and** $l \ge 2$**.** We simultaneously upper bound P_1 and P_2 by replacing b_τ with γb_τ in the upper estimate for P_1. Thus

$$l!(P_1 + P_2) \le \gamma^l (b_v + \cdots + b_t)^l,$$

completing the proof of part (a)(ii).

The proof of part (b) of Theorem 3 is similar to that of part (a). We give a reasonably complete exposition which adapts readily for the proof of Theorem 4 for the hub-authority process. To prove

$$\mathbb{P}(d(v,t) = m+l, d(w,t) = m'+l') = \pi_{l,m}(v,t)\pi_{l',m'}(w,t),$$

we proceed as follows. Assume $v < w$ and let $\boldsymbol{\sigma}(v), \boldsymbol{\sigma}(w)$ be the sequences of transition times for changes in the degree of v, w. Let $\boldsymbol{\sigma}$ be the sequence obtained by merging $\boldsymbol{\sigma}(v), \boldsymbol{\sigma}(w)$. Let $d(v,v) = m$, $d(w,w) = m'$, let $I(t)$ be defined by analogy with (11). The sets J_1, J_2 are also as previously defined (see (26)). Thus $J_1(l+l')$ consists of sequences $\boldsymbol{\sigma}$ of length $l+l'$ where exactly one vertex is hit once at each transition step, and J_2 now includes those sequences where both vertices are hit at the same step.

The random variable $X_v(t)$ gives the number of times edges are incident with vertex v at step $t > v$ (resp. $X_w(t)$). For $t \le w$ define $p'_A(w,t) = 0$ and for $t > w$ let $p'_A(w,t) = p_A(1+O(p_A))$ the correction being for substituting $2|E(t)| - d(v,t)$ for $2|E(t)|$ etc. Given $m(t)$ and $X_v(t) = i$ the distribution of $X_w(t)$ is $\text{Bin}(m(t)-i, p'_A(w,t))$ etc. The value of $\mathbf{Pr}(X_w(t) = k, X_v(t) = i)$ is then given by the generalization of (18)-(22). Conditional on $I(t-1)$, $X_v(t), X_w(t)$ have the following joint probabilities:

$$\mathbf{Pr}(X_v(t) = 0,\ X_w(t) = 0 \mid d(v,t-1) = m+j,\ d(w,t-1) = m'+j') \tag{41}$$
$$= 1 - \left(\frac{\eta(m+j)+\nu}{t} + \frac{\eta(m'+j')+\nu}{t} \right) \left(1+O(\epsilon(t,j+j'))\right), \tag{42}$$

and eg.

$$\begin{aligned}\mathbf{Pr}(X_v(t)=1,\, X_w(t)=0 \mid d(v,t-1)=m+j,\ d(w,t-1)=m'+j')\\ =\frac{\eta(m+j)+\nu}{t}(1+O(\epsilon(t,j+j'))).\end{aligned}$$

Finally, for $k+i\geq 2$,

$$\mathbf{Pr}(X_v(t)=i, X_w(t)=k)\leq \mathbf{Pr}(X_v(t)=i)\mathbf{Pr}(X_w(t)=k)\left(1+O\left(\tfrac{d(v,t)+d(w,t)}{t}\right)\right).$$

Form a sequence $\boldsymbol{\tau}$ from $\boldsymbol{\sigma}$ as follows: Add the symbols v, w in order, except that w is not added if it is already in $\boldsymbol{\sigma}(v)$, (ie. w chose v as a neighbour). Finally let $L=|\boldsymbol{\tau}|$ and add $\tau_{L+1}=t$ to $\boldsymbol{\tau}$. The joint distribution of $d(v,t)$ and $d(w,t)$ can now be derived as follows: For $j\geq 0$, let $n(j,v)$ be the number of occurrences of $\boldsymbol{\sigma}(v)$ in the sub-sequence $(\tau_0,...,\tau_j)$. Define $\mu_j=\eta(m+n(j,v))+\nu$, $\mu'_j=0$ for $\tau_j<w$ and $\mu'_j=\eta(m'+n(j,w))+\nu$ for $\tau_j\geq w$. Following the argument from (27) onwards

$$\Psi_j=\left(\frac{\tau_j}{\tau_{j+1}}\right)^{\mu_j+\mu'_j}\left(1+O(\delta(\tau_j,n(j,v)+n(j,w)))\right),$$

and for $\boldsymbol{\tau}\in J_1$,

$$\Phi_j=\frac{\lambda_j}{\tau_{j+1}}\left(1+O(\epsilon(\tau_{j+1},n(j,v)+n(j,w)))\right),$$

where

$$\lambda_j=\mu_j 1\{\tau_{j+1}\in\boldsymbol{\sigma}(v)\}+\mu'_j 1\{\tau_{j+1}\in\boldsymbol{\sigma}(w)\}.$$

Then similarly to $F(\boldsymbol{\tau})$ in (31)-(33) the probability $H(\boldsymbol{\tau})$ of the sequence of transitions $\boldsymbol{\tau}$ is given by

$$\begin{aligned}H(\boldsymbol{\tau}) &= \left(\prod_{\tau_{j+1}\in\boldsymbol{\sigma}}\frac{\lambda_j}{\tau_{j+1}}\right)\left(\prod_{j=0}^{L}\left(\frac{\tau_j}{\tau_{j+1}}\right)^{\mu_j+\mu'_j}\left(1+O(\delta(\tau_j,n(j,v)+n(j,w)))\right)\right)\\ &= \frac{\Gamma(l+\xi(m))}{\Gamma(\xi(m))}\left(\frac{v}{t}\right)^{\eta\xi(m)}\prod_{i=1}^{l}\frac{\eta\ \sigma_i(v)^{\eta-1}}{t^\eta}\left(1+O(\delta(\sigma_i(v),i+l'))\right)\\ &\quad\times\frac{\Gamma(l'+\xi(m'))}{\Gamma(\xi(m'))}\left(\frac{w}{t}\right)^{\eta\xi(m')}\prod_{i=1}^{l'}\frac{\eta\ \sigma_i(w)^{\eta-1}}{t^\eta}\left(1+O(\delta(\sigma_i(w),i+l))\right).\end{aligned}$$

The deductions following (33) apply to $\sum_{\boldsymbol{\tau}\in J_1}H(\boldsymbol{\tau})$ giving $\mathbb{P}(d(v,t)=m+l, d(w,t)=m'+l', J_1)$. A similar analysis for $\tau\in J_2(l+l')$ completes the proof of part (b) of this theorem. □

We next prove the almost sure maximum degree bound given in Theorem 2(c).

Lemma 1. *If $l^*(v)\geq\omega^2(t/v)^\eta$ then for any $K>0$*

$$\mathbf{Pr}\left(\ \exists\, t\geq v \text{ such that } d(v,t)>K\omega(t/v)^\eta\ \right)=O(v^{-K+2}).$$

Proof Let $\lambda(v,t) = \lfloor K\omega(t/v)^\eta \rfloor$, then $2\lambda + m < l^*(v)$ and so Theorem 3(a)(i),(ii) hold for the range $l \in [\lambda + m, 2\lambda + m]$. As $\binom{l+\xi-1}{l} = O(l^{\xi-1})$, for good processes

$$\begin{aligned}\mathbf{Pr}(d(v,t) \in [m+\lambda, m+2\lambda]) &\leq \lambda O(\lambda^{\xi-1}) \left(\frac{v}{t}\right)^{\eta\xi} e^{-\lambda(1+o(1))(v/t)^\eta} \\ &= \omega^\xi O(t^{-K+1}).\end{aligned}$$

The edge distributions $\boldsymbol{p}, \boldsymbol{q}$ are finite and so at most L edges can be added to v at each step. As $d(v,v) = m < \lambda(v,v) = K \log v$, for $v \geq (\log t)^8$ and t sufficiently large; it is impossible to jump across the $[\lambda + m, 2\lambda + m]$ gap in any one step. The result follows. □

3.3 Theorem 1: Degree sequence

Let $\boldsymbol{D}(l,m) = \{v : d(v,t) = m + l, d(v,v) = m\}$. Theorem 1(a) states that the number of vertices $D(l,m) = |\boldsymbol{D}(l,m)|$ with initial degree m and with degree $m+l$ at step t has expected value $(1 + O(1/\omega))t\alpha p_m n_m(l)$ where

$$n_m(l) = \frac{\Gamma(\xi + 1/\eta)}{\eta\Gamma(\xi)} \frac{\Gamma(l+\xi)}{\Gamma(l+1+\xi+1/\eta)}.$$

For $l = 0, n_m(0) = 1/(1+\eta\xi)$ and for $l \geq 1$, $n_m(l) = \Theta(l^{-(1+1/\eta)})$. Recall from (7) that $n_m(l) = \Theta(l^{-(1+1/\eta)})$. For the values of l considered in Lemma 2, $tn_m(l) = \Omega(t^{1/2})$, and given $\alpha p_m > 0$, the process should have a large number of vertices in $\boldsymbol{D}(l,m)$.

We define an interval $v_1(l) \leq v \leq v_2(l)$ which (usually) contains most of $\boldsymbol{D}(l,m)$. Let $v_1(l) = t/(\omega^2(1_{l=0} + l^{1+1/\eta}))$. Thus for $l \geq 0$, $v_1 = \Theta(tn_m(l)/\omega^2)$.
For $l = 0,1$ let $v_2 = t-1$, for $2 \leq l \leq \omega^3$, let $v_2 = t(1-1/\omega)$, and for $l > \omega^3$ let $v_2 = t(\omega^2/l)^{1/\eta}$.
Let $D_1(l,m) = |\{v_1 \leq v \leq v_2 : v \in \boldsymbol{D}(l,m)\}|$, and $D_2(l,m) = |\{v_2 < v \leq t : v \in \boldsymbol{D}(l,m)\}|$.

Lemma 2. *Let $l^\star = t^{\eta/3}, \eta < 1/2$ and let $l^\star = t^{1/6}/\omega^2, \eta \geq 1/2$. For $l \leq l^\star$,*

(a)

$$\mathbf{E}\ D_1(l,m) = \alpha p_m t\ n_m(l)\left(1 + O\left(\frac{1}{\omega^2}\right)\right).$$

(b)

$$\boldsymbol{Var} D_1(l,m) \leq \mathbf{E}\ D_1(l,m) + (\mathbf{E}\ D_1(l,m))^2 O\left(\frac{1}{\omega^2}\right).$$

(c)(i)

$$\mathbf{E}\ D(l,m) = \alpha p_m t\ n_m(l)\left(1 + O\left(\frac{1}{\omega}\right)\right).$$

(c)(ii)

$$\mathbf{Pr}\left(|D(l,m)-\alpha p_m t n_m(l)| > \frac{\alpha p_m t n_m(l)}{\sqrt{\omega}}\right) = O\left(\frac{1}{\omega}\right).$$

Proof We first prove the results (a)-(c) for processes which are good from step $v_1(l)$ onwards. As $tv^{-K'}$ can be made arbitrarily small for $v \geq v_1(l)$, we can basically ignore bad processes. See the remark at the end of the proof for a formal treatment of this.

Part (a). Recall that $l^*(v) = \min\{t^\eta/\omega^4, v^{1/2}/\omega^4, t^{1/2}(v/t)^\eta/\omega^4\}$ and hence $l^*(v_1(l)) = \min\{t^\eta/\omega^4, t^{1/2}l^{-(1+1/\eta)/2}/\omega^5, t^{1/2}l^{-(1+\eta)}/\omega^{4+2\eta}\}$. By checking on a case by case basis we see that if $l \leq l^\star$ then $l^\star \leq l^*(v_1(l^\star)) \leq l^*(v_1(l))$ and thus Theorem 3(a)(i) holds for $v_1(l)$.

Let $X_v = 1$ if $v \in \boldsymbol{D}_1(l,m)$ and $X_v = 0$ otherwise. Thus

$$\mathbf{E}\ D_1(l,m) = \mathbf{E}\sum_{v_1}^{v_2} X_v = \sum_{v_1}^{v_2} \alpha p_m \pi_{l,m}(v,t).$$

Let $f(v,\epsilon) = (v/t)^{\eta\xi}(1-(v/t)^\eta(1+\epsilon))^l$. From (12)

$$\pi_{l,m}(v,t) = \left(1+O\left(\tfrac{1}{\omega^2}\right)\right)\binom{l+\xi-1}{l} g(v,l),$$

where $f(v,\delta) \leq g(v,l) \leq f(v,-\delta)$ and $\delta = A/\omega^2$, where A is a sufficiently large positive constant.

For $l = 0$ the function $f(v,\epsilon)$ is monotone increasing in v. For $l \geq 1$, and fixed ϵ, $f(v,\epsilon)$ has a unique maximum at $v(l)$ given by the solution of $\xi/(\xi+l) = (1+\epsilon)(v/t)^\eta$. Given ϵ fixed, we can abbreviate $f(v,\epsilon)$ to $f(v)$. As $v_1 \leq v(l) \leq v_2$ and $f(v(l)) = O(l^{-\xi})$, we have

$$\sum_{v=v_1}^{v_2} f(v) = \int_{v_1}^{v_2} f(v)\ dv + O(l^{-\xi}).$$

We note that $\int_0^1 x^{\xi+1/\eta-1}(1-x)^l\ dx = B(\xi+1/\eta, l+1)$. Using the transformation $x = (v/t)^\eta(1+\epsilon)$ we see that

$$\int_{v_1}^{v_2} f(v,\epsilon)\ dv = (1+O(\epsilon))\frac{t}{\eta}B(\xi+1/\eta, l+1) + v_1 O(f(v_1)) + (t-v_2)O(f(v_2)).$$

For $l \geq 1$, $B(\xi+1/\eta, l+1) = \Theta(l^{-(\xi+1/\eta)})$. For $l = 0$ we use $1/(\xi+1/\eta)$. Thus

$$\sum_{v_1}^{v_2} g(v,l) = \frac{t}{\eta}B(\xi+\frac{1}{\eta}, l+1)\left(1+O(\delta)+l^{\xi+1/\eta}\left(O\left(\tfrac{1}{tl^\xi}\right)+O\left(\left(\tfrac{v_1}{t}\right)^{1+\eta\xi}\right)+\left(1-\tfrac{v_2}{t}\right)O\left(f(v_2)\right)\right)\right).$$

For $0 \le l \le \omega^3$, $f(v_2) = O(\omega^{-l})$. For $l \ge \omega^3$, $f(v_2) = O(e^{-\omega^2})$. It follows that $\mathbf{E}\ D_1(l,m) = \alpha p_m t n_m(l)(1 + O(1/\omega^2))$. This completes the proof of part (a) for good processes.

Part (b). Let $X = \sum_{v_1 \le v \le v_2} X_v$ and thus $\mathbf{E}\ X^2 = \mathbf{E}\ X + 2\sum_{v_1 \le v < w \le v_2} \mathbf{E}\ X_w X_v$. From (15) we see that $\mathbf{E}\ X_w X_v = (1 + O(1/\omega^2))\mathbf{E}\ X_w \mathbf{E}\ X_v$ and so

$$\mathbf{E}\ X_w X_v = (1 + O(1/\omega^2))(\alpha p_m)^2 \binom{l+\xi-1}{l}^2 g(v,l)g(w,l).$$

Thus

$$\boldsymbol{Var} D_1(l,m) \le \mathbf{E}\ D_1(l,m) + (\mathbf{E}\ D_1(l,m))^2 O\left(\frac{1}{\omega^2}\right)$$

completing the proof of part (b).

Part (c). We note that $D(l,m) = O(v_1) + D_1(l,m) + D_2(l,m)$.
For part (c)(i) we prove that $\mathbf{E}\ D = \mathbf{E}\ D_1(1 + O(1/\omega))$. For part (c)(ii) we prove that $\mathbf{Pr}(D_2 > \mathbf{E}\ D_1/\omega) = O(1/\omega)$ and apply the Chebychev inequality to D_1.

We defined $v_1 = O(tn_m(l)/\omega^2) = O(\mathbf{E}\ D_1(l,m)/\omega^2)$, which takes care of the $O(v_1)$ term.
For $l = 0, 1$, $D_2(l,m) \le t/\omega = O(\mathbf{E}\ D_1/\omega)$. For $2 \le l \le \omega^3$, from Theorem 3(a)(ii), with $\gamma > 0$ constant,

$$\mathbf{E}\ D_2(l,m) = \frac{t}{\omega} O\left(\frac{\gamma^l l^{\xi-1}}{\omega^l}\right) = \mathbf{E}\ D_1(l,m) O\left(\frac{\gamma^l l^{\xi+1/\eta}}{\omega^l}\right)$$

and thus

$$\mathbf{Pr}(D_2(l,m) \ge \mathbf{E}\ D_1(l,m)/\omega) = O(\gamma^l l^{\xi+1/\eta}/\omega^{l-1}) = O(1/\omega).$$

For $l \ge \omega^3$, $v_2 = t(\omega^2/l)^{1/\eta}$. As $l^*(v_2) \ge l^*(v_1) \ge l = \omega^2(t/v_2)^\eta$, by Lemma 1

$$\mathbf{Pr}(\exists v \ge v_2, t \ge v \text{ such that } d(v,t) \ge l) = O(tv_2^{-K+2}),$$

and $\mathbf{Pr}(D_2(l,m) > 0) = O(tv_2^{-K+2})$. Recalling that $l \le l^\star$, we see that $v_2 \ge t^{2/3}$ and, choosing K sufficiently large, that $\mathbf{E}\ D_2(l,m) = o(\mathbf{E}\ D_1/\omega^2)$.

Completing the proof. Say a process is bad if it was not good at/after $v_1(l)$. For vertices $v \ge v_1(l)$ we can choose K' so that $v^{-K'} = o(n_m(l)/\omega^2)$. By construction $v_1(l) = O(tn_m(l)/\omega^2)$, and thus

$$\mathbf{E}\ (D(l,m),\ \text{bad}\ ) \le tO(v^{-K'}) = O(\mathbf{E}\ (D(l,m),\ \text{good}\ )/\omega^2),$$

completing this part of the proof, and also the proof of Theorem 1(a). □

4 Hub-authority web-graph process

In the Hub-Authority model of Kleinberg [11] the initial vertex of an inserted edge is chosen preferentially according to out-degree, and the terminal vertex is chosen preferentially according to in-degree. We define a generalized hub-authority process based on a mixture of uar and preferential attachment. Results on the number of vertices of a given directed degree (d^-, d^+) and the distribution of the degree of a given vertex are obtained as extensions of the proof of Theorems 1, 2 for the undirected process. This approach seems easier than deriving the expected degree sequence using recurrence relations as in [8] which seems rather difficult in general, although this method was used by Bollobás, Borgs, Chayes and Riordan in [4].

4.1 Definition of the process

Independently at each step either a NEW-OUT procedure is followed (with probability α), a NEW-IN procedure (with probability γ) or an OLD procedure (with probability β) where $\alpha, \beta, \gamma \geq 0$ constant and $\alpha + \beta + \gamma = 1$.

NEW-OUT **procedure.** Chosen with probability α at step t. A new vertex $v_t = t$ is added, and directs $m^+(t)$ edges from itself to $G(t-1)$. The distribution $\boldsymbol{p}^+$ of $m^+(t)$ is finite. The terminal vertex of each edge is selected independently from $G(t-1) = (V(t-1),\ E(t-1))$ by a mixture of uar and preferential attachment on in-degree. Thus each new edge independently selects vertex $w \in V(t-1)$ as terminal vertex with probability

$$p_A(w,t) = A_1 \frac{d^-(w,t-1)}{|E(t-1)|} + A_2 \frac{1}{|V(t-1)|}. \tag{43}$$

NEW-IN **procedure.** Chosen with probability γ at step t. A new vertex $v_t = t$ is added, and directs $m^-(t)$ edges towards itself from $G(t-1)$. The distribution $\boldsymbol{p}^-$ of $m^-(t)$ is finite. The initial vertex of each edge is selected independently from $G(t-1)$ according to a mixture of uar and preferential attachment on out-degree. Thus each new edge selects vertex w as initial vertex with probability

$$p_D(w,t) = D_1 \frac{d^+(w,t-1)}{|E(t-1)|} + D_2 \frac{1}{|V(t-1)|}. \tag{44}$$

OLD **procedure.** Chosen with probability β at step t. The distribution $\boldsymbol{q}$ of the number of inserted edges $M(t)$ is finite. Each edge makes an independent choice of initial and terminal vertex. The initial vertex of the edge is chosen using an out-degree/uar mixture with probability $p_B(w,t)$ similar in definition to $p_D(w,t)$ in (44). The terminal vertex is chosen using an in-degree/uar mixture with probability $p_C(w,t)$, similar in definition to p_A in (43).

We note that it is straightforward to allow a NEW vertex to choose a mixture of in- and out-edges with distribution $\boldsymbol{p}(m^-, m^+)$ and the theorems given generalize naturally to this by replacing the procedures NEW-IN, NEW-OUT by a joint procedure.

4.2 Hub-authority process: Statement of results

Define

$$\eta^- = \frac{\alpha \overline{m}^+ A_1 + \beta \overline{M} C_1}{\alpha \overline{m}^+ + \gamma \overline{m}^- + \beta \overline{M}} \qquad \nu^- = \frac{\alpha \overline{m}^+ A_2 + \beta \overline{M} C_2}{\alpha + \gamma} \qquad \xi^-(v) = m^-(v) + \nu^-/\eta^- \quad (45)$$

$$\eta^+ = \frac{\gamma \overline{m}^- D_1 + \beta \overline{M} B_1}{\alpha \overline{m}^+ + \gamma \overline{m}^- + \beta \overline{M}} \qquad \nu^+ = \frac{\gamma \overline{m}^- D_2 + \beta \overline{M} B_2}{\alpha + \gamma} \qquad \xi^+(v) = m^+(v) + \nu^+/\eta^+. \quad (46)$$

The parameters (45) are for the distribution of in-degree, and (46) for the distribution of out-degree. At first glance the form of η^-, ν^- etc. may look unusual, but, in the hub-authority model the in-degree of existing vertices depends in part on the average out-degree $\overline{m}^+$ of NEW-OUT vertices. The parameter η^- is the limiting ratio of the expected number of edges whose terminal vertex was chosen by preferential attachment, to the expected number of edges of the process.

When we refer to definitions and results of the undirected process, it is on the understanding that the appropriate parameter substitutions have been made. For example, in the case of $d^-(v,t)$ we use $(m^-, \xi^-, \eta^-, \nu^-, l^{*-})$ instead of (m, ξ, η, ν, l^*) and similarly $\pi_{l,m^-}(v,t;\eta^-,\nu^-)$ is defined by analogy with (12). For $m^- > 0$ we define $n^-_m(l)$ in the obvious way as $n_m(l)$ of (5) with all parameters θ replaced by θ^-. Similar definitions hold for out-degree.

We always assume $\alpha + \gamma > 0$ so that the expected size of the vertex set grows with t. In general it makes sense to assume that $\xi^-(v),\ \xi^+(v) > 0$. In this way both the in-degree and out-degree of any vertex v can grow over time, irrespective of their initial values. However we do not exclude special cases where this does not happen. Thus for $m^- = 0$ and $\nu^- = 0$ define $\pi_{0,0}(v,t;\eta^-,0) = 1$ and $\pi_{l,0}(v,t;\eta^-,0) = 0,\ l \geq 1$. We extend the definition of $n_{m^-}(l)$ for $m^- = 0$ as follows: If $\nu^- > 0$ define $n_0(l)$ by (5) with $\xi^-(0) = \nu^-/\eta^-$. If $\nu^- = 0$ define $n_0(0) = 1, n_0(l) = 0,\ l \geq 1$.

We shall show, subject to restrictions on v, r, s, that the distribution of in-degree $(d^-(v,t) = r)$ and out-degree $(d^+(v,t) = s)$ of a given vertex v are asymptotically independent and the joint distribution is the product of negative binomial distributions for the in- and out-degrees. The asymptotic independence of the marginal distributions means that the expected number of vertices of a given in-degree (resp. out-degree) is given by the equivalent of Theorem 1.

Theorem 4. *(a) (i) Suppose $\alpha > 0$. The marginal distribution of in-degree $d^-(v,t)$ is given by the equivalent of Theorem 2 (a)-(c), (resp. $\gamma > 0$, out-degree $d^+(v,t)$).*

(ii) The joint distribution is (asymptotically in t) the product of the marginal distributions. In particular, if Theorem 2(a) holds for the marginals then

$$\mathbf{Pr}(d^-(v,t) = m^- + r, d^+(v,t) = m^+ + s \mid d^-(v,v) = m^-, d^+(v,v) = m^+) = \tag{47}$$
$$=\pi_{r,m^-}(v,t;\eta^-,\nu^-)\pi_{s,m^+}(v,t;\eta^+,\nu^+) + O(v^{-K}).$$

(b) Let $D_t^-(r,m^-)$ denote the number of vertices with initial in-degree m^- and in-degree $r+m^-$ at step t. Assuming $r \le l^{\star -}$ of Theorem 1, then with probability $1 - O(1/\omega)$

$$D_t^-(r,m^-) = p(m^-)n_{m^-}(r)t \ \left(1 + O\left(\frac{1}{\sqrt{\omega}}\right)\right),$$

where for $m^- \ge 1$, $p(m^-) = \gamma p^-_{m^-}$ and for $m^- = 0$, $p(m^-) = \alpha$. Similar results hold for $D_t^+(s,m^+)$.

Theorem 4 is proved in Section 4.3. Bollobás, Borgs, Chayes and Riordan [4] also considered a directed scale-free model generalized to include uar choices. They established the value of the power law parameter for the case where eg. $r \to \infty$, s constant, and other results similar to Theorem 4(b).

For $\xi^-(m^-), \xi^+(m^+) > 0$ define

$$n(r,s;m^-,m^+) = \binom{r+\xi^- - 1}{r}\binom{s+\xi^+-1}{s}\int_0^1 x^{\eta^-\xi^- + \eta^+\xi^+}(1-x^{\eta^-})^r(1-x^{\eta^+})^s dx. \tag{48}$$

Theorem 5. *Assume $\xi^-(m^-), \xi^+(m^+) > 0$. Let $v_1 = tn(r,s)/\omega^2$. Let $D_t(r,s;m^-,m^+)$ be the number of vertices with initial degree (m^-,m^+), and in-degree $r+m^-$, out-degree $s+m^+$ at the end of step t. Let (r,s) be such that $r \le l^{*-}(v_1)/\omega^2$ and $s \le l^{*+}(v_1)/\omega^2$. Then*

$$\mathbf{E}\ D_t(r,s;m^-,m^+) = tp(m^-,m^+)n(r,s;m^-,m^+)\left(1 + O\left(\tfrac{1}{\omega}\right)\right),$$

where $p(m^-,m^+) = (\gamma p^-_{m^-})1(m^- > 0) + (\alpha p^+_{m^+})1(m^+ > 0)$.

This theorem is proved in Section 4.4. For certain values of r,s the value of $n(r,s)$ can be written asymptotically as

$$n(r,s;m^-,m^+) = \Theta(1)r^{-x^-}s^{-x^+},$$

where the parameters x^-, x^+ depend on the relative sizes of r and s and change as s increases from $s = O(r^{\eta^+/\eta^-})$ to $s = o(r^{\eta^+/\eta^-})$. The significance of $s^{1/\eta^+} = r^{1/\eta^-}$ can be seen informally as follows: The expected in-degree of vertex v is

$$\mathbf{E}\ (d^-(v,t) \mid d^-(v,v) = m^-) \sim (m^-(v) + \nu^-/\eta^-)(t/v)^{\eta^-}.$$

Define $v(r) = t/r^{1/\eta^-}$, then the vertex $v(r)$ has expected in-degree proportional to r at step t, and expected out-degree s proportional to $(t/v(r))^{\eta^+} = r^{\eta^+/\eta^-}$.

Theorem 6. *If $r \to \infty$ and $s = O(r^{\eta^+/\eta^-})$ then*

$$n(r, s; m^-, m^+) = \Theta(1) r^{-\left(1+\frac{\eta^+}{\eta^-}\xi^+ + \frac{1}{\eta^-}\right)} s^{-\left(1-\xi^+\right)}. \tag{49}$$

If $s \to \infty$ and $r = O(s^{\eta^-/\eta^+})$ then

$$n(r, s; m^-, m^+) = \Theta(1) r^{-\left(1-\xi^-\right)} s^{-\left(1+\frac{\eta^-}{\eta^+}\xi^- + \frac{1}{\eta^+}\right)}. \tag{50}$$

This theorem is proved in Section 4.5. If $s = \Theta(r^{\eta^+/\eta^-})$ expressions (49), (50) are equivalent to $n(r, s) = \Theta(1) r^{-(1+(\eta^+ + 1)/\eta^-)}$. The precise values of the multiplicative constants can be obtained from (48) and Lemma 3 of Section 4.5.

4.3 Proof of Theorem 4

To prove Theorem 4, we adapt the proofs of Section 3 to the hub-authority process. The part that requires explanation is the generalization of the proof of Theorem 3.

Note that a NEW-OUT event at step t results in the increase of the in-degree of $G(t-1)$ etc. The variables $X^-(t)$ (resp. $X^+(t)$) for the alteration of in-degree and out-degree of any existing vertex at step t are distributed as follows:

$$\begin{aligned} X^-(t) &\sim 1_\alpha(t)\mathrm{Bin}(m^+(t), p_A(t)) + 1_\beta(t)\mathrm{Bin}(M(t), p_C(t)) \\ X^+(t) &\sim 1_\gamma(t)\mathrm{Bin}(m^-(t), p_D(t)) + 1_\beta(t)\mathrm{Bin}(M(t), p_B(t)). \end{aligned}$$

The transition probabilities for a given vertex v of degree $(m^- + j^-, m^+ + j^+)$ at step $t+1$ are as follows. For $\mathbf{Pr}(X^-(t+1) = 0,\ X^+(t+1) = 0)$ the calculations corresponding to those made for (21) are

$$\begin{aligned} &\mathbf{Pr}(X^-(t+1) = 0,\ X^+(t+1) = 0 \mid d^-(v,t) = m^- + j^-,\ d^+(v,t) = m^+ + j^+) \\ &\quad = \mathbf{E}\ \left(1_\alpha(1-p_A)^{m^+(t+1)} + 1_\beta[(1-p_B)(1-p_C)]^{M(t+1)} + 1_\gamma(1-p_D)^{m^-(t+1)}\right) \\ &\quad = 1 - \left(\frac{\eta^-(m^- + j^-) + \nu^-}{t} + \frac{\eta^+(m^+ + j^+) + \nu^+}{t}\right)\left(1+\epsilon(j^-, j^+, t)\right), \end{aligned}$$

where $\eta^\pm, \nu^\pm$ are given by (45), (46) and

$$\epsilon(j^-, j^+, t) = O\left(\sqrt{\log t/t} + [(m^- + j^-) + (m^+ + j^+)]/t\right).$$

The expression for $\mathbf{Pr}(X^-(t+1)=1, X^+(t+1)=0)$ etc. is similar to (41)-(42) of Theorem 3. Higher order transitions ($X^- + X^+ \geq 2$) are bounded by

$$\mathbf{Pr}(X^-=i, X^+=k) \leq \mathbf{Pr}(X^-=i)\mathbf{Pr}(X^+=k)(1+o(1)),$$

which is the correct order of magnitude for transitions in J_2.

Let $d^-(v,v) = 1_\gamma(v)m^-$, $d^-(v,t) = 1_\gamma(v)m^- + l^-$ etc. Let $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_{l^-})$, $\boldsymbol{\tau} = (\tau_1, ..., \tau_{l^+})$ be the transition times of the in- and out-degrees of vertex v. The calculations for the marginal distributions are identical with those made in Theorem 3. In particular, provided $\nu^- > 0$ the fact that $m^-(v) = 0$ does not affect the validity of the estimates for $F(\boldsymbol{\sigma})$ in (27)-(34) etc. The joint in- and out-degree distribution of a given vertex v is obtained in a similar way to the joint distribution of a pair of vertices v, w in the undirected case. Straightforward generalizations also give the joint distribution of pairs of vertices v, w.

4.4 Proof of Theorem 5

Write

$$I(r,s) = \int_0^1 x^{\eta^-\xi^- + \eta^+\xi^+}(1-x^{\eta^-})^r(1-x^{\eta^+})^s dx, \tag{51}$$

then $n(r,s;m^-,m^+) = \binom{r+\xi^--1}{r}\binom{s+\xi^+-1}{s} I(r,s)$.

The proof follows that of Theorem 1 in Section 3.3. Let $D(r,s) = D(r,s;m^-,m^+)$ denote the number of vertices of $G(t)$ with in-degree $r+m^-$, out-degree $s+m^+$, and initial in-degree m^- and out-degree m^+. Define v_1, v_2 by analogy with Lemma 2; thus eg. $v_1 = tn(r,s)/\omega^2$. Let $D_1(r,s)$ be the restriction of $D(r,s)$ to $v_1 \leq v \leq v_2$. The conditions $r \leq l^{*-}(v_1)/\omega^2$, $s \leq l^{*+}(v_1)/\omega^2$ ensure that the marginal degree distributions of Theorem 4(a) are valid for $v \geq v_1$, and

$$\begin{aligned}
\mathbf{E}\, D(r,s) &= \sum_{v=1}^{t} \mathbf{Pr}(d^-(v,t) = m^- + r, d^+(v,t) = m^+ + s; m^-, m^+) \\
&= {\scriptstyle (1+O(1/\omega))} \sum_{v=v_1}^{v_2} \mathbf{Pr}(d^-(v,t) = m^- + r; m^-)\mathbf{Pr}(d^+(v,t) = m^+ + s; m^+) \\
&= (1+O(1/\omega))tp(m^-,m^+)n(r,s;m^-,m^+). \qquad (52)
\end{aligned}$$

Recall that $l^*(v)$ is defined in Theorem 2 by $l^*(v) = \min\{t^\eta/\omega^4,\ v^{1/2}/\omega^4,\ t^{1/2}(v/t)^\eta/\omega^4\}$. By fixing the relationship $s = f(r)$ the range of validity of Theorem 5 can be expressed if so desired as a function of the single variable r. For example, when $s = \Theta(r^{\eta^+/\eta^-})$, then by

Theorem 6 $n(r,s) = \Theta(1)r^{-x}$ where $x = 1 + (\eta^+ + 1)/\eta^-$. Using these results we find that we require

$$r \le \min\left\{ \frac{t^{\eta^-}}{\omega^A}, \frac{t^{1/2}r^{-x/2}}{\omega^A}, \frac{t^{1/2}r^{-\eta^- x}}{\omega^A} \right\},$$

where $A = A(\eta^-, \eta^+)$ is a positive constant.

4.5 Proof of Theorem 6

The proof of Theorem 6 follows from the lemma below, which gives a good approximation for $I(r,s)$ of (51), and hence $n(r,s;m^-,m^+)$.

Lemma 3. *Let $a = \eta^-, b = \eta^+, c = \eta^-\xi^- + \eta^+\xi^+$. Assuming $r \to \infty$ and $s = Ar^{\eta^+/\eta^-}$ where A is at most a large positive constant, then $I(r,s)$ of (51) can be written*

$$I(r,s) = (1+o(1))\frac{1}{a}\left(\frac{1}{r}\right)^{\frac{c+1}{a}} \int_0^\infty w^{\frac{c+1}{a}-1} e^{-\left(w + Aw^{b/a}\right)} dw.$$

If $A \to 0$ the integral on the right hand side tends to $C_A = \Gamma((c+1)/a)$, and for any $A = \Theta(1)$ it has positive constant value $C_A \le \Gamma((c+1)/a)$. It follows that

$$I(r,s) \sim C_A/(ar^{(c+1)/a}).$$

Proof Let $x_0 = (\log^2 r/r)^{1/a}$, then $(1-x_0^a)^r \le \exp(-\log^2 r)$. For sufficiently large r, the contribution to the integral $I(r,s)$ from $x \ge x_0$ is $o(1/r^{(c+1)/a})$. For $x \le x_0$,

$$\begin{aligned}(1-x^a)^r(1-x^b)^s &= \exp -\left(rx^a + sx^b + O(rx^{2a}) + O(sx^{2b})\right) \\ &= \exp -\left(rx^a\left(1 + O(\log^2 r/r)\right) + sx^b\left(1 + O(\log^2 r/r)^{b/a}\right)\right).\end{aligned}$$

Using the transformation $w = rx^a$, the result follows. □

5 Undirected hypergraph process

It seems reasonable, in the light of copy-based models such as those in [12] and [13], to consider processes where bunches of edges are added in groups at each step. One simple approach is to add hyper-edges of varying sizes from two up to some upper limit r. If we so wish, we can view these hyper-edges as cliques, thus increasing local clustering of the process as befits small world behavior.

We consider an undirected process as defined in Section 2.1, except now the edge sizes are also random. Let $\boldsymbol{\pi} = (\pi_2, ..., \pi_r)$ be the distribution of edge sizes $j = 2, ..., r$ for the NEW procedure, and $\boldsymbol{\rho} = (\rho_2, ..., \rho_R)$ the distribution of edge sizes $j = 2, ..., R$ for the OLD procedure. For finite $\boldsymbol{\pi}, \boldsymbol{\rho}$, let $\overline{\pi} = \sum j\pi_j$ and let $\overline{\rho} = \sum j\rho_j$. Various assumptions are possible for the insertion of NEW and OLD edges. For NEW edges we assume the terminal vertices are chosen independently with probability p_A. For OLD edges, we assume that some vertex designated as initial is chosen with probability p_B and the terminal vertices are chosen independently with probability p_C.

The distribution of $X(t)$ the number of edges selecting vertex v at step t is given by

$$X(t) \sim 1_\alpha(t) \left(\sum_{i=1}^{m(t)} \mathrm{Bin}([R_i^\pi - 1], p_A(t))\right) + 1_\beta(t) \left(\mathrm{Bin}(M(t), p_B(t)) + \sum_{i=1}^{M(t)} \mathrm{Bin}([R_i^\rho - 1], p_C(t))\right)$$

where $R_i^\pi \sim \boldsymbol{\pi}$ is a random variable giving the size of edge e_i, $i = 1, ..., m(t)$ in the NEW procedure, and $R_i^\rho \sim \boldsymbol{\rho}$ gives the size of edge e_i, $i = 1, ..., M(t)$ in the OLD procedure.

By taking $\mathbf{E}\, X(t)$ within the set $I(t)$ of good histories, we see that the parameters η, ν of the process are as follows:

$$\eta = \frac{\alpha\overline{m}(\overline{\pi} - 1)A_1 + \beta\overline{M}(B_1 + (\overline{\rho} - 1)C_1)}{\alpha\overline{m}\,\overline{\pi} + \beta\overline{M}\overline{\rho}} \qquad \nu = \frac{\alpha\overline{m}(\overline{\pi} - 1)A_2 + \beta\overline{M}(B_2 + (\overline{\rho} - 1)C_2)}{\alpha}.$$

The details of the proof and the results are similar to those of Theorem 1, 2.

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