

Edge disjoint Hamilton cycles in sparse random graphs of minimum degree at least k

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Abstract

Let $G_{n,m,k}$ denote the space of simple graphs with n vertices, m edges and minimum degree at least k , each graph G being equiprobable. Let G have property \mathcal{A}_k if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. For $k \geq 3$, \mathcal{A}_k occurs in $G_{n,m,k}$ with probability tending to 1 as $n \rightarrow \infty$, when $2m \geq c_k n$ for some suitable constant c_k .

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1 Introduction

Denote by $G_{n,m,k}$ the space of simple graphs with vertex set $[n] = \{1, 2, \dots, n\}$, m edges and minimum degree at least k , each graph being equiprobable. We say a graph G has property \mathcal{A}_k , if G contains $\lfloor (k-1)/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is even, a further edge disjoint matching of size $\lfloor n/2 \rfloor$. We prove the following theorem.

Theorem 1 *Let $k \geq 3$. There exists a constant $c_k \leq 2(k+1)^3$ such that if $2m \geq c_k n$ then **whp**¹ any $G \in G_{n,m,k}$ has property \mathcal{A}_k .*

In the paper *Hamilton cycles in random graphs of minimal degree at least k* [BFF], Bollobás, Fenner and Frieze establish a sharp threshold for a stronger property \mathcal{A}_k^* , in the case where $m/n \rightarrow \infty$. A graph G has property \mathcal{A}_k^* , if G contains $\lfloor k/2 \rfloor$ edge disjoint Hamilton cycles, and, if k is odd, a further edge disjoint matching of size $\lfloor n/2 \rfloor$.

Theorem 2 (BFF) *Let $2m = n \left(\frac{\log n}{k+1} + k \log \log n + d_n \right)$. If $G \in G_{n,m,k}$, then*

$$\lim_{n \rightarrow \infty} \Pr(G \in \mathcal{A}_k^*) = \begin{cases} 0 & \text{if } d_n \rightarrow -\infty, \\ e^{-\theta_k(d)} & \text{if } d_n \rightarrow d, \\ 1 & \text{if } d_n \rightarrow +\infty, \end{cases} \quad \text{slowly}$$

where

$$\theta_k(d) = \frac{e^{-(k+1)d}}{(k+1)! \{(k-1)!\}^{k+1} (k+1)^{k(k+1)}}.$$

The primary obstruction to property \mathcal{A}_k^* in $G_{n,m,k}$ is the presence of k -spiders. A k -spider is $k+1$ vertices of degree k having a common neighbour. In [BFF] it was shown that

$$\lim_{n \rightarrow \infty} \Pr(G \in G_{n,m,k} \text{ has a } k\text{-spider}) = 1 - e^{-\theta_k(d)}.$$

When $2m = cn$, where c is constant, k -spiders occur **whp** in $G_{n,m,k}$. Thus **whp** the property \mathcal{A}_k^* does not occur, and property \mathcal{A}_k is best possible.

2 Models for the space $G_{n,m,k}$

For any graph G in $G_{n,m,k}$ there are $m!$ ways to order, and 2^m ways to orient the edges, to give $2^m m!$ sequences of vertex labels of length $2m$. The set $\mathcal{S}(n, 2m, k)$ of sequences arising in this manner has the uniform measure induced by $G_{n,m,k}$. Let $\mathcal{M}(n, 2m, k)$ be

¹**whp** with high probability. With probability tending to 1 as $n \rightarrow \infty$.

the space of equiprobable sequences $(a_i : a_i \in [n], i = 1, \dots, 2m)$ specifying which of n labelled boxes contains each of $2m$ labelled balls; with the condition that the minimum occupancy of any box is at least k . Each element of \mathcal{M} defines a multigraph with vertex set $[n]$. The set \mathcal{S} is the subset of \mathcal{M} whose underlying graphs are simple.

Denote by $\mathcal{O}(n, 2m, k)$ the space of sequences $(b_j : b_j \in [2m], j = 1, \dots, n)$, giving the possible occupancies of the n boxes (the degrees of the vertices) arising from sequences in $\mathcal{M}(n, 2m, k)$, with the derived probability measure. A useful method of obtaining results about $\mathcal{O}(n, 2m, k)$ is to consider a larger space $\mathcal{P}(n, \lambda, k)$ in which each of the n boxes has independent occupancy given by a truncated Poisson random variable X with parameters λ and k . The space $\mathcal{O}(n, 2m, k)$ is obtained from $\mathcal{P}(n, \lambda, k)$ by conditioning on $\sum_{i=1}^n X_i = 2m$, as is explained in [BFU]. Let

$$\Pr(X = j) = \frac{\lambda^j e^{-\lambda}}{j! \beta(\lambda, k)} \quad j \in \{k, k+1, \dots\}$$

where

$$\beta(\lambda, k) = 1 - e^{-\lambda} \left(1 + \lambda + \dots + \frac{\lambda^{k-1}}{(k-1)!} \right).$$

It is natural to choose λ so as to maximize the probability of the conditioning event $\{\sum_i X_i = 2m\}$. This can be approximately achieved by making $\mathbf{E}X$ equal to c , the average vertex degree. Explicitly, we choose

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \tag{1}$$

so that $\mathbf{E}X = c$. The properties of (1) are given in Lemma A1 of the appendix.

If we consider an event \mathcal{E} in $G_{n,m,k}$ which specifies the degree sequence $\mathbf{d}(S)$ of some subset S of vertices, the set of multigraph sequences with $\mathbf{d}(S)$ in $\mathcal{M}(n, 2m, k)$ is well defined and corresponds to well defined events in $\mathcal{O}(n, 2m, k)$ and $\mathcal{P}(n, \lambda, k)$. For such an *occupancy event*, \mathcal{E} , we can regard \mathcal{E} as being defined in each of the spaces under consideration, rather than just in $G_{n,m,k}$.

Lemma 3 *Let k be a fixed positive integer, and let $m = cn/2$, where $c \geq k$. Let \mathcal{E} be an occupancy event, then for sufficiently large n ,*

(i)

$$\Pr(\mathcal{E}; \mathcal{O}(n, 2m, k)) \leq (1 + o(1)) \sqrt{2\pi cn} \Pr(\mathcal{E}; \mathcal{P}(n, \lambda, k))$$

(ii)

$$\Pr(\mathcal{E}; G_{n,m,k}) \leq e^{O(c^2)} \Pr(\mathcal{E}; \mathcal{O}(n, 2m, k)).$$

For convenience we denote generic expressions of the form $O(\sqrt{c}) e^{O(c^2)}$ by $A(c)$.

We will frequently use the model $\mathcal{P}(n, \lambda, k)$ to estimate the probability of an event \mathcal{E} , that a set S of vertices of a graph G in $G_{n,m,k}$ has degree sum T . In the model $\mathcal{P}(n, \lambda, k)$, the probability that a set S of s boxes (with occupancy $t_i : i = 1, \dots, s$) has total occupancy T is

$$\begin{aligned} \Pr(t_1 + \dots + t_s = T; \mathcal{P}(n, \lambda, k)) &= \sum_{\substack{t_1 + \dots + t_s = T \\ t_i \geq k}} \left(\prod_{i=1}^s \frac{\lambda^{t_i} e^{-\lambda}}{t_i! \beta(\lambda, k)} \right) \\ &= \frac{\lambda^T e^{-\lambda s}}{[\beta(\lambda, k)]^s} \frac{1}{T!} \sum_{\substack{t_1 + \dots + t_s = T \\ t_i \geq k}} \binom{T}{t_1 \dots t_s} \\ &\leq \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda, k)]^s}. \end{aligned} \quad (2)$$

Lemma 4 *Let k be a fixed positive integer, and let $2m = cn$ where $c \geq k$. For sufficiently large n ,*

(i)

$$\frac{|G_{n,m,k}|}{|G_{n,m}|} \geq e^{-O(c^2)} \left[\left(\frac{c}{\lambda e} \right)^c e^\lambda \beta(\lambda, k) \right]^n.$$

(ii) *Let $D = \frac{kc^{k-1}}{(k-1)!} e^{-(c-k)} e^{k^2/(c-k)}$, then*

$$\Pr(\delta(G) \geq k; G_{n,m}) \geq e^{-nD}.$$

The proofs of Lemmas 3, 4 are provided in the appendix.

Although $|G_{n,m,k}|$ is bounded below by an exponentially small multiple of $|G_{n,m}|$, the constant D in the exponent also is small. For example, when $c = c_k$, $k = 3$, then $D = 1.4 \times 10^{-50}$. In particular, note that if $c \geq c_k$ then

$$D \leq e^{-8c/9}. \quad (3)$$

This allows us to prove results concerning $G_{n,m,k}$ directly in $G_{n,m}$ in Lemma 8 by using the general estimate

$$\Pr(\mathcal{E}; G_{n,m,k}) = \Pr(\mathcal{E} \mid \delta(G) \geq k) \leq \frac{\Pr(\mathcal{E}; G_{n,m})}{\Pr(\delta(G) \geq k; G_{n,m})}. \quad (4)$$

We will frequently use the following crude upper bound for the conditional probability that a sequence in $\mathcal{M}(n, 2m, k)$ corresponds to a multigraph in which there are q edges between A and B given that the sets of vertices A, B have degree a, b respectively, namely

$$\binom{m}{q} \left(\frac{a}{2m} \frac{b}{2m} \right)^q 2^{q\delta(A,B)} \quad (5)$$

where $\delta(A, B) = 0$ if $A = B$ and 1 if A and B are disjoint. We note that $\binom{m}{q} \leq (me/q)^q$ and that the unconstrained maximum of $(me/q)^q$ with respect to q occurs at $q = me$.

3 Relevant properties of $G_{n,m,k}$

Let $P_{y_h} = y_0y_1\dots y_h$ be a longest path starting at y_0 (y_0 path) in G . A Pósa rotation $P_{y_h} \rightarrow P_{y_{i+1}}$, [Po],[Bo] gives the path $P_{y_{i+1}} = y_0y_1\dots y_iy_hy_{h-1}\dots y_{i+1}$ formed from P_{y_h} by adding edge y_hy_i and erasing edge y_iy_{i+1} . We call y_hy_i , y_iy_{i+1} the *transformation edges* and $y_{i+1}y_{i+2}$ the *adjacent edge* of the rotation.

Pósa rotations of a longest x_0 path $Px_h = x_0Px_h$ with x_0 fixed, define a *rotation subgraph* $\mathcal{R} = \mathcal{R}(x_0)$ of G , as follows. Initially $\mathcal{R} = x_{h-1}x_h$, where x_h is an *active* endpoint. Perform all possible rotations based on x_h due to edges x_hx_i , adding the transformation and adjacent edges of each rotation to \mathcal{R} . Each x_{i+1} is now an active endpoint, whereas x_h is now *passive*. The final graph \mathcal{R}^* is not necessarily unique, but we consider it to be fully explored, in the following sense. Let x be an active endpoint with $P_x = x_0\dots yz\dots x$ and where there is an edge xy in G . If z is a passive endpoint we add xy to \mathcal{R} and consider this a transformation edge, else we add the transformation and adjacent edges corresponding to the rotation $P_x \rightarrow P_z$.

At any stage, we define a subgraph \mathcal{T} of \mathcal{R} which includes only transformation edges, where initially, $\mathcal{T} = x_h$.

Lemma 5 *Let G be a non-Hamiltonian graph of minimum degree at least 3 and let P_0 be a longest path in G . One of the following two possibilities must hold:*

- (a) *G is connected and the rotation subgraph \mathcal{R} of P_0 contains at least two cycles.*
- (b) *G is not connected and the vertices of P_0 form a single component.*

Proof We consider a final graph $\mathcal{R} = \mathcal{R}^*$. Let U be the set of endpoints of P_0 obtained while constructing \mathcal{R}^* . We note that in \mathcal{R} each vertex of U is on a cycle. This follows because $U \subseteq V(\mathcal{T})$ and \mathcal{T} has minimum degree at least 2.

Suppose that \mathcal{T} consists of exactly one cycle C . Note that $U \subseteq V(C)$ and every edge of C is incident with a vertex of U . There are two cases to consider.

Case I. $N(U) \cap U = \emptyset$. The vertices of U alternate with vertices of $N(U)$ on C .

Thus the cycle C is of length $2|U|$. Each vertex of U in C has at least one further vertex of $N(U)$ attached as a pendant leaf of C . This follows as the minimum degree in \mathcal{R} of any vertex in U is at least 3, but there is only one cycle. Thus $|U \cup N(U)| \geq 3|U|$, contradicting the Pósa condition that $|N(U)| \leq 2|U| - 1$.

Case II. Two vertices of U are adjacent on C .

We claim that every vertex on the cycle is an endpoint. We proceed inductively.

Orient C , and let $(u, v) \in C$, $u, v \in U$. Consider P_u . If $P_u = x_0 \dots vx \dots u$, then $x \in U$. Hence $(v, x) \in C$ and $v, x \in U$. If $P_u = x_0 \dots vu$ then $P_u = x_0 \dots a_1 b_1 \dots a_2 b_2 \dots vu$ where there are edges $\{a_1, u\}$, $\{a_2, u\}$ and paths $ua_1 b_1$, $ua_2 b_2$. At best $C = \dots b_1 a_1 uv \dots$, and there is a chordal path $ua_2 b_2$ of C as $b_2 \in U$. This contradicts the unicyclicity assumption.

Suppose now that $U = V(C)$. Consider the initial longest path $P_0 = x_0 x_1 \dots x_h$. Let $b \in U$ be the first occurrence of an endpoint vertex in P_0 beyond x_0 . Thus either $P_0 = x_0 b \dots x_h$ or $P_0 = x_0 Qab \dots x_h$.

In the first case there exists a sequence of transformations P_{x_h}, \dots, P_w such that $x_0 w$ is an edge of G . This implies there is a cycle through the vertices of P_0 . As we suppose G is non-Hamiltonian, G cannot be connected else P_0 is not a longest path. Thus (b) holds here.

In the second case there must be an endpoint w , an edge wa and a transformation $P_w \rightarrow P_b$ using wa . As $x_0 Qa$, the initial segment of P_0 is never broken, the vertex $a \notin C$. We conclude that \mathcal{R} is not unicyclic as wab is a chordal path of C . \square

Lemma 6 *Let $s_0 = n (3/c)^3 e^{-(2k+12)}$ and $S \subset [n]$, $|S| < s_0$. There exists a c_k such that for $c \geq c_k$ **whp** no $G \in G_{n,m,k}$ has any subset of vertices S which induces at least $3|S|/2$ edges.*

Proof Let $|S| = s$, fix the degree sum T of S . The expected number of vertex sets S inducing at least $3s/2$ edges is at most

$$\gamma(s) = A(c) \sqrt{n} \binom{n}{s} \sum_{T \geq ks} \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda, k)]^s} \binom{m}{3s/2} \left(\frac{T}{2m} \right)^{3s},$$

where the right hand side of this expression follows from (2) and (5) and Lemma 3. Thus, as $c - k \leq \lambda \leq c$ we can apply (10) as follows.

$$\gamma(s) \leq A(c) \sqrt{n} \left(\frac{ne}{s} \right)^s \frac{e^{ks}}{[\beta(\lambda, k)]^s} \left(\frac{e}{3scn} \right)^{3s/2} \left[e^{-cs} \sum_{T \geq ks} \frac{(cs)^T}{T!} T^{3s} \right]$$

$$\begin{aligned}
&\leq A(c)\sqrt{n} \frac{e^{(k+1)s}}{[\beta(\lambda, k)]^s} \left(\frac{n}{s}\right)^s \left(\frac{e}{3scn}\right)^{3s/2} (cse^{3/k})^{3s} \\
&\leq A(c)\sqrt{n} \left(\frac{s}{n} \frac{c^3}{3^3} e^{(2k+5+18/k+2k^2/(c-k))}\right)^{s/2},
\end{aligned}$$

where any constants have been absorbed into $A(c)$. Provided $k \geq 3$ and $c > 2k^2 + k$,

$$\sum_{4 \leq s \leq s_0} \gamma(s) = o\left(\frac{1}{n}\right)$$

□

Lemma 7 *Let $S \subseteq [n]$, $|S| = s$, $1 \leq s \leq n$. There exists a constant c_k such that if $c \geq c_k$ then **whp** no $G \in G_{n,m,k}$ has any set of vertices S such that*

- (i) *the degree sum T of S satisfies $ks \leq T \leq cs/4k$,*
- (ii) *$|N(S)| \leq k|S|$,*
- (iii) *the subgraph F consisting of the edges induced by S and by $S \times N(S)$, is connected and contains at least two cycles.*

Proof Let the pair $(S, N(S))$ satisfy the conditions of (i),(ii) and (iii). Let $T = 2q+p$ where there are q edges induced by S and p edges induced by $S \times N(S)$. Denote $|N(S)|$ by r . Let R be the degree sum of $N(S)$, where $R \geq \max\{p, kr\}$. Because F is connected and contains at least two cycles we have that

$$p + q \geq r + s + 1,$$

and thus

$$T \geq q + r + s + 1.$$

For fixed T, r, s the expected number of pairs $(S, N(S))$ satisfying conditions (i),(ii),(iii) is at most $\gamma(r, s, T)$ where from (5),

$$\begin{aligned}
\gamma(r, s, T) &= A(c)\sqrt{n} \binom{n}{s} \binom{n}{r} \frac{(\lambda s)^T}{T!} \frac{e^{-\lambda s}}{[\beta(\lambda, k)]^s} \left(\sum_{R \geq p} \frac{(\lambda r)^R}{R!} \frac{e^{-\lambda r}}{[\beta(\lambda, k)]^r} \right) \\
&\quad \times \binom{m}{p} \binom{m}{q} \left[\binom{T}{2m} \binom{T}{2m} \right]^q \left[\binom{T}{2m} \binom{R}{2m} \right]^p.
\end{aligned}$$

Thus, as $c - k \leq \lambda \leq c$ an upper bound on $\gamma(r, s, T)$ is given by

$$\frac{A(c)\sqrt{n}}{[\beta(\lambda, k)]^s} \left(\frac{s}{n}\right)^{T-(r+s+q)} e^{(k+1)s-cs+T} \left(\frac{ces}{2q}\right)^q \left(\frac{e}{p}\right)^p \left(\frac{se^{k+1}}{\beta(\lambda, k)r}\right)^r \left[e^{-cr} \sum_{R \geq p} \frac{(cr)^R}{R!} R^p \right].$$

From (10) of Lemma A3,

$$e^{-cr} \sum_{R \geq p} \frac{(cr)^R}{R!} R^p \leq 2(crc)^p,$$

which is monotone increasing in r . The function $(se^{k+1}/(\beta(\lambda, k) r))^r$ is monotone increasing in $r = |N(S)|$ over the range $0 \leq |N(S)| \leq k|S|$, and the constrained maximum occurs at $r = ks$ giving

$$\gamma(r, s, T) \leq \frac{A(c)}{[\beta(\lambda, k)]^s} \left(\frac{s}{\sqrt{n}}\right)^{e^{(k+1)^2 s}} \frac{e^{(k+1)^2 s}}{k^{ks}} e^{-cs} e^T \left(\frac{ces}{2q}\right)^q \left(\frac{kce^2 s}{p}\right)^p.$$

Now, $p = T - 2q$ and $1 - x > \exp\{-x/(1-x)\}$ so, $p^{-p} \leq T^{-T} (eT)^{2q}$ and thus

$$\begin{aligned} e^T \left(\frac{ces}{2q}\right)^q \left(\frac{kce^2 s}{p}\right)^p &\leq \left(\frac{kce^3 s}{T}\right)^T \left(\frac{T^2 e}{2k^2 ce^2 s q}\right)^q \\ &\leq \exp\left\{\frac{T^2}{2k^2 ce^2 s}\right\} \left(kce^2 s \frac{e}{T}\right)^T \\ &\leq \left[\exp\left\{\frac{1}{8k^3 e^2}\right\} (4k^2 e^3)\right]^{\frac{cs}{4k}}. \end{aligned}$$

Thus provided $k \geq 3$ and $c \geq 2(k+1)^3$ say,

$$\begin{aligned} \sum_{r,s,T} \gamma(r, s, T) &\leq \frac{cA(c)}{\sqrt{n}} \sum_{s=1}^n s^3 \left[\frac{(5k^2 e^3)^{c/4k}}{k^k} e^{-c+(k+1)^2 + \frac{k^2}{c-k}}\right]^s \\ &\leq \frac{cA(c)}{\sqrt{n}} \sum_{s=1}^n s^3 e^{-cs/5}. \end{aligned}$$

□

Lemma 8 Let $\alpha = \frac{1}{\sqrt{2(k+1)}}$. Let $s_0 = n(3/c)^3 \exp\{-(2k+12)\}$, $s_1 = \alpha n$.

Let $S \subset [n]$, $|S| = s$, $s_0/(k+1) \leq s \leq s_1$.

If $c \geq c_k$ then **whp** no $G \in G_{n,m,k}$ has any S such that $|N(S)| \leq k|S|$.

Proof Let \mathcal{E} denote the event that there exists a set S , such that $|N(S)| \leq k|S|$.

We work in $G_{n,p}$, where $p = c/n$. We assume that $\Gamma_S = G[S \cup N(S)]$ is connected or we can consider each connected subgraph separately. There are $(k+1)s - 1$ edges in a spanning tree of Γ_S and suppose there are q other edges incident with S in Γ_S . The expected number $\gamma(q)$ of such Γ_S is at most,

$$\gamma(q) \leq \binom{n}{s} \binom{n-s}{ks} ((k+1)s)^{(k+1)s-2} \binom{\binom{s}{2} + ks^2}{q}$$

$$\begin{aligned}
& \times \left(\frac{c}{n}\right)^{q+(k+1)s-1} \left(1 - \frac{c}{n}\right)^{s(n-(k+1)s)+\binom{s}{2}+ks^2-(q+(k+1)s-1)} \\
& \leq A(c) \frac{n}{s^2} \left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}e^{-c}}{k^k}\right)^s \left(\frac{s^2(k+\frac{1}{2})e^{1+c/n}c}{nq}\right)^q e^{(c-2k)s^2/(2n)}.
\end{aligned}$$

Now, $(xe/q)^q \leq e^x$, so

$$\begin{aligned}
\max_q \gamma(q) & \leq A(c) \frac{n}{s^2} \left(\frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k} e^{-c} e^{\frac{(k+1)cs}{n}}\right)^s \\
& \leq A(c) \exp \left\{ -cs \left(\left(1 - \frac{1}{c} \log \frac{c^{k+1}e^{k+1}(k+1)^{k+1}}{k^k}\right) - \frac{(k+1)s}{n} \right) \right\} \quad (6) \\
& \leq A(c) e^{-2Dn}.
\end{aligned}$$

Provided $\lambda < 1$ the function $(1-\lambda)s - \frac{(k+1)s^2}{n}$ has an unconstrained maximum at $s = \frac{(1-\lambda)n}{2(k+1)}$. For the value of λ given in (6) above, this is in the range (s_0, s_1) . The right hand side of (6) is minimized at s_0 , and the final inequality follows from (3). Thus from Lemmas 3 and 4, and (4)

$$\Pr(\mathcal{E}; G_{n,m,k}) \leq \sqrt{ne}^{-Dn}$$

□

Lemma 9 For $c \geq c_k$, $G \in G_{n,m,k}$ satisfies the following conditions **whp**.

- (i) If $S \subset [n]$, $\alpha n \leq |S| \leq (1-\alpha)n$ then there are at least $4k|S|$ edges from S to $[n] - S$,
- (ii) If $L(3c) = \{e \in E(G) : e \text{ is incident with a vertex of degree at least } 3c\}$ then $|L(3c)| \leq ne^{-c/6}$
- (iii) Let $L(k) = \{e \in E(G) : e \text{ is incident with a vertex of degree } k\}$ then $|L(k)| \leq ne^{-c/6}$.

Proof In all three cases we estimate the probability of failure in $G_{n,m}$ or $G_{n,p}$, $p = c/n$ and show it is at most e^{-2nD} . This estimate can then be inflated by $O(n^{1/2}e^{nD})$ as was done in Lemma 8.

(i) We assume w.l.o.g. that $s = |S| \leq n/2$. In $G_{n,p}$ the number of edges between S and $[n] - S$ has binomial distribution $B(s(n-s), p)$. This has mean $s(n-s)p$ and if $4ks = (1-\epsilon)s(n-s)p$ then $\epsilon \geq 2/3$ (using $c \geq 2(k+1)^3$ and $s \leq n/2$.) Applying the Chernoff-Hoeffding bound

$$\Pr(B(N, \theta) \leq (1-\epsilon)N\theta) \leq e^{-\epsilon^2 N\theta/3}$$

we obtain

$$\begin{aligned}
\Pr((i) \text{ fails in } G_{n,p}) &\leq \sum_{s=\alpha n}^{n/2} \binom{n}{s} e^{-sc/9} \\
&\leq \sum_{s=\alpha n}^{n/2} \left(\frac{ne}{s} e^{-c/9} \right)^s \\
&\leq n(e\sqrt{2}(k+1)e^{-2(k+1)^3/9})^{n/(\sqrt{2}(k+1))} \\
&\leq e^{-2nD}.
\end{aligned}$$

(ii) The probability that an edge $e \in G_{n,m}$ is incident with a vertex of degree at least $3c$ is at most 3^{-c} . Thus $\mathbf{E}(|L(3c)|) \leq 3^{-c}n$ in $G_{n,m}$. Changing one edge of $G_{n,m}$ changes $|L(3c)|$ by at most $12c$. Applying the Azuma-Hoeffding martingale tail inequality we get

$$\begin{aligned}
\Pr((ii) \text{ fails in } G_{n,m}) &\leq \exp\{-n(e^{-c/6} - 3^{-c})^2/(144c^2)\} \\
&\leq \exp -2nD.
\end{aligned}$$

(iii) The probability that edge $e \in G_{n,m}$ is incident with a vertex of degree k is at most $\exp -2c/3$. We proceed as in (ii). \square

4 The proof of Theorem 1

We will write k as $2l + 1$ if k is odd, and as $k = 2l + 2$ otherwise. The property \mathcal{A}_k requires the existence of l edge disjoint Hamilton cycles $H_i : i = 1, \dots, l$, and, if k is even, a further edge disjoint (near) perfect matching H_0 .

We prove the existence **whp** of these structures in a sequential manner. Initially $i = 0$, and the set of excluded edges $\mathcal{Q}(0)$ is empty. If k is even, we first prove **whp** the existence of a Hamilton cycle by the methods described below, and use the edges of this cycle to obtain a (near) perfect matching H_0 . The edges of the matching H_0 are added to \mathcal{Q} . At the start of iteration $i = 1, \dots, l$, the set $\mathcal{Q}(i)$ contains those edges to be excluded from the cycle H_i , by virtue of appearing in H_0, \dots, H_{i-1} . Thus

$$\mathcal{Q}(i) = \cup_{j=0}^{i-1} E(H_j).$$

To prove the existence of H_i , we follow the method of Fenner and Frieze [FF]. A set \mathcal{T} of edges of G is said to be *deletable* if

D(a) \mathcal{T} is not incident with any vertex of degree k or degree at least $3c$,

D(b) \mathcal{T} avoids a specified longest path $P_0 = x_0 P x_h$ in $G - \mathcal{Q}$,

D(c) \mathcal{T} avoids the specified set \mathcal{Q} ,

D(d) \mathcal{T} is a matching.

Let $\mathcal{N}(G)$ be the set of edges of G which \mathcal{T} must avoid in order to satisfy the conditions D(a),(b),(c) above and let $H = G - \mathcal{T} - \mathcal{Q}(i)$. Let $END(x_0; H)$ be a rotation endpoint set of the fixed longest path $x_0 P x_h$ in the subgraph H , and $\mathcal{R}(x_0)$ the associated rotation subgraph.

Lemma 10 *Let \mathcal{B} be the subset of graphs in $G_{n,m,k}$ which satisfy the conditions of Lemma's 6, 7, 8 and 9, then*

$$(i) |\mathcal{B}| = (1 - o(1))|G_{n,m,k}|,$$

$$(ii) \text{ If } G \in \mathcal{B} \text{ then } |END(x_0; H)| \geq \alpha n,$$

(iii) $G \in \mathcal{B}$ implies H is connected.

Proof (i) This is a consequence of Lemmas 6-9.

(ii) Let $END(x_0; H) = S$. We assume $|S| < \alpha n$.

At the start of iteration $i = 0, 1, \dots, l$ the degree in $\mathcal{Q}(i)$ of any vertex $v \in [n]$ is $2(i - 1) + 1\{i \geq \delta \text{ and } k = 2l + 2\}$. We note that $|N_G(S)| < k|S|$, for if $|N_G(S)| \geq k|S|$ then $|N_{G-\mathcal{Q}(i)}(S)| \geq 3|S|$. We delete at most a matching from S in $G - \mathcal{Q}$, so this implies that $|N_H(S)| \geq 2|S|$ in contradiction of the Pósa condition. By Lemma 8, $|S| \leq s_0/(k + 1)$.

If (b) of Lemma 5 holds then $S = P_0$ and S induces a connected subgraph of minimum degree at least 3. This contradicts the condition of Lemma 6.

Assume now that (a) of Lemma 5 holds. As $\mathcal{R}(x_0)$ is a connected graph it satisfies Lemma 7(iii), and $|N(S)| < ks$ satisfies 7(ii). Thus we conclude that T the degree of S satisfies $T > cs/4k$. Let $|N(S)| = \theta s < ks$. By Lemma 6, the total number of edges in $S \cup N(S)$ satisfies

$$p + q \leq \frac{3}{2}(1 + \theta)s,$$

and the total number of edges in S satisfies

$$q \leq \frac{3}{2}s.$$

A simple optimization shows that $T = 2q + p$ is maximized at $(p, q) = (3\theta s/2, 3s/2)$. Thus $T \leq 3s + \frac{3}{2}\theta s$ so that $T < \frac{3}{2}(k+2)s$. However, as $T > \frac{cs}{4k}$, this implies $c < 6k(k+2)$, and contradicts the assumption that $c \geq c_k$.

(iii) Starting with any vertex v , we see that both $\mathcal{R}(v)$ and P_0 , are connected subgraphs of H , containing at least αn vertices by (ii). The connectivity of H follows from Lemma 9. \square

Let \mathcal{E} be the subset of $G_{n,m,k}$ which does not have property \mathcal{A}_k . We will apply the edge colouring argument of Fenner and Frieze [FF] in an inductive manner to the set $\mathcal{E} \cap \mathcal{B}$ to prove that $\Pr(\mathcal{E}) \rightarrow 0$

Let \mathcal{T} be a deletable set of edges of G of size $t = \lceil \log n \rceil$ avoiding the set $\mathcal{N}(G)$ of size s , and let αn be a lower bound on $|END(x_0)|$. By transforming P_0 in H using Pósa rotations there are at least $(\alpha n)^2/2$ longest paths aPb in H with distinct endpoint sets $\{a, b\}$. Thus if $G \in \mathcal{E}$ at least $(\alpha n)^2/2$ non-edges must be avoided when replacing a subgraph \mathcal{T}' , to form a graph $G' = G - \mathcal{T} + \mathcal{T}'$, $G' \in \mathcal{E}$. We will call such a replacement subgraph *addable*.

Let η be a lower bound on the number of ways of selecting a deletable \mathcal{T} from G . Then

$$\begin{aligned} \eta &\geq \frac{1}{t!} \prod_{j=0}^{t-1} (m - s - 6cj) \\ &\geq \frac{1}{t!} m^t \exp \left\{ -\frac{t(s + 6ct)}{m - (s + 6ct)} \right\}, \end{aligned}$$

where

$$s = |\mathcal{N}(G)| \leq \left(\frac{1}{2}(k-1) + 2e^{-c/6} \right) n.$$

Let μ be an upper bound on the number of ways of choosing an addable edge set \mathcal{T}' , then

$$\mu \leq \binom{\binom{n}{2} - \binom{\alpha n}{2} - (m-t)}{t} \leq \frac{1}{t!} \binom{n}{2}^t e^{-\alpha^2 t}.$$

The argument of [FF] states that,

$$\Pr(\mathcal{E} \cap \mathcal{B}) \leq (1 + o(1)) \frac{\mu |G_{n,m-t,k}|}{\eta |G_{n,m,k}|}.$$

Now

$$\begin{aligned} \frac{|G_{n,m-t,k}|}{|G_{n,m,k}|} &= \frac{\binom{\binom{n}{2}}{\binom{m-t}{2}} \Pr(\delta \geq k \text{ in } G_{n,m-t})}{\binom{\binom{n}{2}}{\binom{m}{2}} \Pr(\delta \geq k \text{ in } G_{n,m})} \\ &\leq (1 + o(1)) \left(\frac{m}{\binom{n}{2}} \right)^t \end{aligned}$$

since the probability $\Pr(\delta \geq k \text{ in } G_{n,m-t})$ is non-decreasing as $t \rightarrow 0$.

We now find that

$$\Pr(\mathcal{E}) \leq O(1) \exp \left\{ -O \left(t \left(\alpha^2 - \frac{s}{m-s} \right) \right) \right\}.$$

We require that

$$\alpha = \frac{1}{\sqrt{2}(k+1)} > \sqrt{\frac{(k-1) + 4e^{-c/6}}{c - ((k-1) + 4e^{-c/6})}},$$

which is satisfied when $c \geq c_k = 2(k+1)^3$. The value of α is the same as in Lemma 8.

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6 Appendix

Numerical calculations show that even for moderate values of c , the value of λ rapidly converges to c provided k is small. Similarly $\beta(\lambda, k)$ tends rapidly to 1. For example, $1 - 10^{-10} \leq \beta(30, 3) \leq 1$. Thus we can effectively treat λ and c as equal, and ignore $\beta(\lambda, k)$. The properties of $\beta(\lambda, k)$ are given in the following lemma.

Lemma A 1 *Let $\lambda = \lambda(c)$ be defined by*

$$\lambda \frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = c, \quad (7)$$

where

$$\begin{aligned} \beta(\lambda, 0) &= 1, \\ \beta(\lambda, k) &= 1 - e^{-\lambda} \left(1 + \lambda + \dots + \frac{\lambda^{k-1}}{(k-1)!} \right), \quad k \geq 1. \end{aligned}$$

For $c > k$ the function $\lambda(c)$ is well defined and

- (i) $c - k < \lambda < c$,
- (ii) (a) $\frac{\beta(\lambda, k-1)}{\beta(\lambda, k)}$ is a monotone decreasing function of λ , tending to 1 as $\lambda \rightarrow \infty$,
- (b) $\frac{\beta(\lambda, k-1)}{\beta(\lambda, k)}$ is a monotone increasing function of k ,
- (iii) $\left(\frac{c}{\lambda e}\right) e^{\lambda/c} > 1$,
- (iv) (a) $\beta(\lambda, k)$ is a monotone increasing function of λ tending to 1 as $\lambda \rightarrow \infty$.
- (b) $\frac{1}{\beta(\lambda, k)} \leq e^{\frac{k^2}{c-k}}$.
- (c) $\beta(\lambda, k) \geq e^{-D}$ where $D = \frac{kc^{k-1}}{(k-1)!} e^{-(c-k)} e^{k^2/(c-k)}$.

Proof (i),(ii) We note that

$$\frac{\beta(\lambda, k-1)}{\beta(\lambda, k)} = 1 + \frac{k}{\lambda} \frac{1}{\left(1 + \frac{\lambda}{k+1} + \frac{\lambda^2}{(k+1)(k+2)} + \dots + \frac{\lambda^i}{(k+1)^{(i)}} + \dots\right)}. \quad (8)$$

(iii) For simplicity denote $\beta(\lambda, i)$ by β_i , and $\frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda}$ by ℓ then $\beta_{k-1} = \beta_k + \ell$ and

$$\left(\frac{c}{\lambda e}\right) e^{\lambda/c} = \left(e^{\left(\frac{\beta_k}{\beta_{k-1}} - 1\right)} \frac{\beta_{k-1}}{\beta_k}\right)$$

$$\begin{aligned}
&= \left(e^{-\frac{\ell}{\beta_k + \ell}} \right) \left(1 + \frac{\ell}{\beta_k} \right) \\
&> \left(1 - \frac{\ell}{\beta_k + \ell} \right) \left(1 + \frac{\ell}{\beta_k} \right) \\
&= 1.
\end{aligned}$$

(iv)(b) This comes from (8) by iterating

$$\beta(\lambda, j-1) \leq \left(1 + \frac{j}{\lambda} \right) \beta(\lambda, j).$$

(iv)(c) This follows by applying (iv)(b) to the right hand side of

$$\beta(\lambda, k) \geq \exp \left\{ - \frac{(1 + \lambda + \dots + \frac{\lambda^{k-1}}{(k-1)!}) e^{-\lambda}}{\beta(\lambda, k)} \right\}.$$

□

Proof of Lemma 3

Let $X = X_i$ be a truncated Poisson random variable with parameters λ, k giving the occupancy of cell i , then $\mathbf{E}X = c$ where $cn = 2m$. Conditioning on $\sum_{i=1}^n X_i = 2m$ in $\mathcal{P}(n, \lambda, k)$, we obtain $\mathcal{O}(n, 2m, k)$. Specifically, if $\mathbf{x} \in \mathcal{P}(n, \lambda, k)$, $\mathbf{x} = (X_1, \dots, X_n)$ and $\sum_{i=1}^n X_i = 2m$ so that $\mathbf{x} \in \mathcal{O}(n, 2m, k)$, then

$$\mathbf{Pr}(\mathbf{x}; \mathcal{O}(n, 2m, k)) = \mathbf{Pr}([\mathbf{x} \mid \sum X_i = 2m]; \mathcal{P}(n, \lambda, k)).$$

This is a generalization of the result that a multinomial random variable may be obtained from independent Poisson random variables by conditioning on their sum; and the details are described in, for example, [BFU]. Thus

$$\mathbf{Pr}(\mathcal{E}; \mathcal{O}(n, 2m, k)) \leq \frac{\mathbf{Pr}(\mathcal{E}; \mathcal{P}(n, \lambda, k))}{\mathbf{Pr}(\sum_{i=1}^n X_i = 2m; \mathcal{P}(n, \lambda, k))}.$$

By the Local Limit Theorem, (see [BFU] and [Du]) $\mathbf{Pr}(\sum_{i=1}^n X_i = 2m)$ is asymptotic to $1/\sqrt{2\pi\sigma^2 n}$, where $\sigma^2 = \mathbf{Var}(X)$ is given by

$$\sigma^2 = \lambda^2 \frac{\beta(\lambda, k-2)}{\beta(\lambda, k)} - c^2 + c, \tag{9}$$

and by (7) and Lemma A1(ii)(b) we see that $\sigma^2 < c$.

An element $\mathbf{d} = (d_i : i = 1, \dots, n)$ of $\mathcal{O}(n, 2m, k)$, induces a set of configuration multi-graphs $M(\mathbf{d}) \subset \mathcal{M}(n, 2m, k)$. The expected number of loops and multiple edges in a configuration multigraph arising from an element of $\mathcal{O}(n, 2m, k)$, is a function of the

degree sequence \mathbf{d} . The maximum occupancy of any box in $\mathcal{O}(n, 2m, k)$ is $o(\log n)$ with probability of the complementary event $n^{-O(\log \log n)}$. Conditioning on maximum degree $o(\log n)$ and using the methods given in [Bo], the probability there are no loops or multiple edges in such a configuration multigraph is asymptotic to $\exp\{-\zeta/2 - \zeta^2/4\}$, where

$$\zeta = \frac{1}{2m} \sum_{i=1}^n d_i(d_i - 1).$$

In $\mathcal{P}(n, \lambda, k)$, the random variable ζ is the sum of independent random variables and is sharply concentrated with expected value

$$\mathbf{E}\zeta = \lambda \frac{\beta(\lambda, k - 2)}{\beta(\lambda, k - 1)}.$$

Fix $\epsilon > 0$, small. Then with probability $1 - e^{-\Omega(\epsilon^2 n)}$, we have $(1 - \epsilon)\mathbf{E}\zeta \leq \zeta \leq (1 + \epsilon)\mathbf{E}\zeta$ in $\mathcal{P}(n, \lambda, k)$ and $\mathcal{O}(n, 2m, k)$. If \mathcal{S} is the subset of $\mathcal{M}(n, 2m, k)$ corresponding to $G_{n,m,k}$, then

$$\begin{aligned} \Pr(\mathcal{S}; \mathcal{M}(n, 2m, k)) &= o(1) + \sum_{\mathbf{d}} (1 + o(1)) \exp\{-(\zeta(\mathbf{d})/2 + \zeta(\mathbf{d})^2/4) \Pr(M(\mathbf{d}); \mathcal{M}(n, 2m, k))\} \\ &= (1 + O(\epsilon)) \exp\{-(\mathbf{E}\zeta/2 + (\mathbf{E}\zeta)^2/4)\}. \end{aligned}$$

Now

$$\Pr(\mathcal{E}; G_{n,m,k}) = \frac{\Pr(\mathcal{E} \cap \mathcal{S}; \mathcal{M}(n, 2m, k))}{\Pr(\mathcal{S}; \mathcal{M}(n, 2m, k))} \leq \frac{\Pr(\mathcal{E}; \mathcal{O}(n, 2m, k))}{\Pr(\mathcal{S}; \mathcal{M}(n, 2m, k))}.$$

□

The following lemma, and its proof are due to B. Pittel [Pi], who uses this approach in, for example, [PW] and [Pi1]. The proof technique uses the Local Limit Theorem (see [Gn], [Du]) to avoid the direct application of a saddle point method. The origins of this technique can be traced back to A. I. Kinchin [Ki]. The use of the Local Limit Theorem in conjunction with generating functions for problems of this type was championed by V. F. Kolchin (see [Ko] for a wide ranging discussion).

Lemma A 2 (Pi)

$$\frac{|\mathcal{M}(n, 2m, k)|}{|\mathcal{M}(n, 2m)|} = (1 + o(1)) \sqrt{\frac{c}{\sigma^2}} \left[\left(\frac{c}{\lambda e} \right)^c e^{\lambda \beta(\lambda, k)} \right]^n.$$

Proof Let $f(z) = \sum_{j \geq k} \frac{z^j}{j!}$ so that $f(z) = e^z \beta(z, k)$, and let $[z^t]g(z)$ denote the coefficient of z^t in the power series of $g(z)$.

$$\begin{aligned} \frac{|\mathcal{M}(n, 2m, k)|}{|\mathcal{M}(n, 2m)|} &= \sum_{\substack{b_1, \dots, b_n \geq k \\ b_1 + \dots + b_n = 2m}} \binom{2m}{b_1 \cdots b_n} \left(\frac{1}{n} \right)^{2m} \\ &= \frac{(2m)!}{n^{2m}} [z^{2m}](f(z))^n. \end{aligned}$$

Let $x > 0$, so that

$$[z^{2m}](f(z))^n = \frac{(f(x))^n}{x^{2m}} [z^{2m}] \left(\frac{f(zx)}{f(x)} \right)^n.$$

Let $Y(x)$ be a random variable chosen so that

$$\mathbf{E}z^Y = \frac{f(zx)}{f(x)},$$

where, such a Y exists as $\left. \frac{f(zx)}{f(x)} \right|_{z=1} = 1$. Let Y_1, \dots, Y_n be independent copies of Y . Then

$$\begin{aligned} [z^{2m}] \left(\frac{f(zx)}{f(x)} \right)^n &= [z^{2m}] \mathbf{E}(z^{Y_1 + \dots + Y_n}) \\ &= \mathbf{Pr}(Y_1 + \dots + Y_n = 2m). \end{aligned}$$

Now

$$\mathbf{E}Y = \left. \frac{d}{dz} \mathbf{E}z^Y \right|_{z=1} = \frac{xf'(x)}{f(x)} = x \frac{\beta(x, k-1)}{\beta(x, k)},$$

so that, if we choose $x = \lambda$, then $\mathbf{E}Y = c$. Similarly, $\mathbf{Var}(Y) = \sigma^2$ is given by (9). By the Local Limit Theorem,

$$\mathbf{Pr}(Y_1 + \dots + Y_n = 2m) \sim \frac{1}{\sqrt{2\pi\sigma^2n}}.$$

Hence

$$\frac{|\mathcal{M}(n, 2m, k)|}{|\mathcal{M}(n, 2m)|} = \frac{1 + o(1)}{\sqrt{2\pi\sigma^2n}} \frac{(2m)!}{n^{2m}} \frac{(f(\lambda))^n}{\lambda^{2m}},$$

and the result follows. \square

Proof of Lemma 4

(i) We have for $j = 0$ and k that

$$|G_{n,2m,j}| 2^m m! = (1 + o(1)) e^{-\theta_j/2 - \theta_j^2/4} |\mathcal{M}(n, 2m, j)|,$$

where $\theta_0 = c$, and $\theta_k = \lambda\beta(\lambda, k-2)/\beta(\lambda, k-1)$ was shown in the proof of Lemma 3.

(ii) Apply Lemma A1 (i),(iii) and (iv). \square

Lemma A 3 *Let t, b be integer, where $t \geq b \geq 0$, and let $a > 0$, then*

$$\sum_{T \geq t} T^b \frac{a^T}{T!} \leq 2e^a (ae^{b/t})^b. \quad (10)$$

Proof Suppose $T \geq t \geq b$, then

$$\frac{T^b}{(T)_b} = \frac{T^b(T-b)!}{T!} \leq 2 \left(1 - \frac{b}{T}\right)^{T-b} e^b \leq 2 \exp\left\{\frac{b^2}{T}\right\}.$$

Thus

$$\begin{aligned} \sum_{T \geq t} T^b \frac{a^T}{T!} &\leq \sum_{T \geq t} 2e^{b^2/T} \left((T)_b \frac{a^T}{T!} \right) \\ &\leq 2e^{b^2/t} a^b e^a. \end{aligned}$$

□