The Computational Complexity of the Regular Resolution Width Problem

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Resolution is a well-known refutation system for sets of clauses Σ (the *axioms*), or, equivalently, propositional formulae in CNF. It operates with the single resolution rule.

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A resolution refutation $\mathscr{R}_\Sigma$ of Σ is a sequence of clauses $C_1, \ldots, C_s$ s.t.

- each C_k is either in Σ or is the resolvent of two clauses C_i and C_j with $i, j < k$, and
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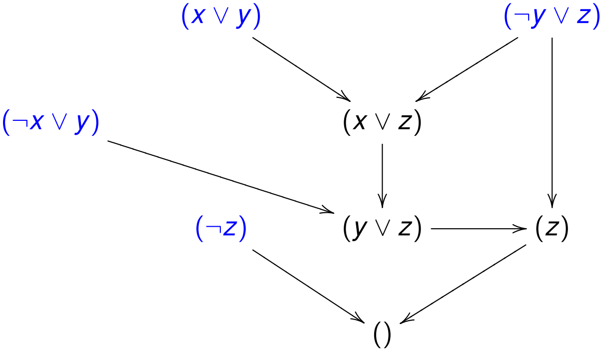
Resolution is known to be both sound and complete. We may associate with $\mathscr{R}_\Sigma$ a digraph with vertices $C_1, \ldots, C_s$ and directed edges from two clauses to their resolvent . This digraph is acyclic with sink $C_s :=$ () and all sources axioms in Σ.

Consider $\Sigma := \{(x \vee y), (\neg y \vee z), (\neg x \vee y), (\neg z)\}$. The following $\mathscr{R}_\Sigma$ is a sample resolution refutation of Σ.



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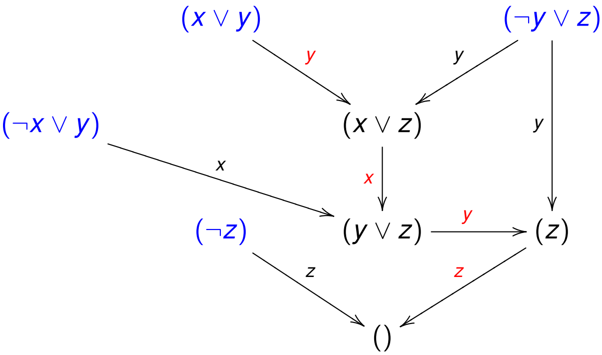
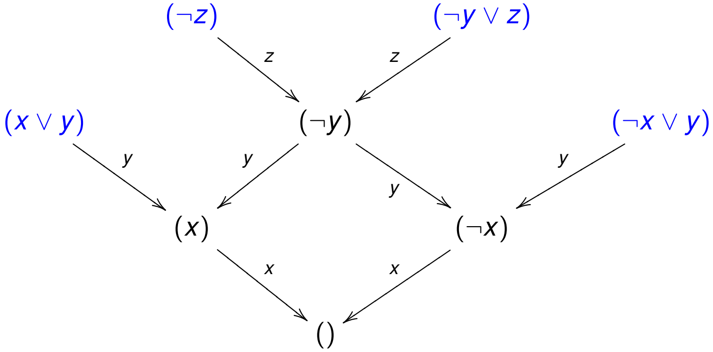
Each of these is provably exponentially weaker than resolution.

Let us return to our $\mathscr{R}_\Sigma$. It is neither tree-like, Davis-Puttnam nor regular.



But the following $\mathscr{R}'_\Sigma$ depicts an alternative resolution refutation of Σ that is regular.



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The *resolution width problem* is defined to have

- Input: a set of clauses Σ and a positive integer k.
- Question: does Σ have a resolution refutation of width $\leq k$?

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- Input: two structures $\mathcal{A}, \mathcal{B}$ and a positive integer k.
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k-pairs of pebbles. Spoiler plays on $\mathcal{A}$; Duplicator plays on $\mathcal{B}$, trying to keep partial homomorphism.

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This *KAI pebble game problem* is primitive in that its Exptime-completeness is derived directly by reduction from an APspace $=$ Exptime Turing Machine.

The *KAI pebble game* is played on a structure $\mathcal{B} := (X, S, R, t)$ where X is the domain, $S \subseteq R$ is a set of initial positions, $R \subseteq X^3$ is a set of rules and $t \in X$ is a target node. Play begins with pebbles on the positions in S; Players I and II move alternately, each aiming to move some pebble to the target t. A pebble may be move from x to x' only if there exists a y s.t. x and y are pebbled, but x' is not, and $(y, x', x) \in R$.

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The *KAI pebble game problem*,

- Input: a structure $\mathcal{B} := (X, S, R, t)$.
- Question: does Player I have a winning strategy in the KAI game?

is Exptime-complete.

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The condition of acyclicity ensures Pspace $=$ APtime membership by limiting the length of the KAI pebble game, because a certain pebble may not return to a node that it has previously visited.
In fact the acyclic KAI pebble game problem is Pspace-complete.

We introduce the following *acyclic existential E-F pebble game problem*.

- Input: two structures $\mathcal{A}, \mathcal{B}$ and a positive integer k.
- Question: does Spoiler have a winning strategy in the existential k-pebble E-F game on $(\mathcal{A}, \mathcal{B})$ in which no element may be pebbled more than once?

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The final piece of the jigsaw is a reduction from the acyclic existential E-F pebble game problem to the regular resolution width problem, thus proving the Pspace-completeness of the latter.

KAI pebble game

$\downarrow \leq$

existential E-F pebble game

$\downarrow \leq$

resolution width problem

Exptime-complete

acyclic KAI pebble game

$\downarrow \leq$

acyclic existential E-F pebble game

$\downarrow \leq$

regular resolution width problem

Pspace-complete

KAI pebble game

↓ $\leq$

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regular resolution width problem

Pspace-complete

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The reduction from the KAI pebble game problem to the existential E-F pebble game problem does not preserve this parameter; the number of pebbles involved in the E-F game is independent of the number of initial pebbles in the given KAI game instance.

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