

On the synchronization delay of complete local automata

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London Algorithmic Workshop (LAW)
King's College London, 7-8 February 2007

Local automata

Informally, in a non-deterministic local automaton, a fixed amount of input symbols determine the current state.

This strong synchronization property makes them helpful to construct codes for constrained channels.

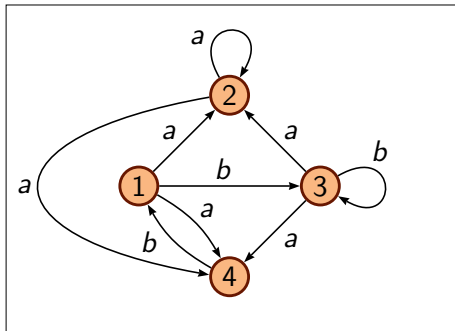
We consider automata whose states are both initial and final and with a strongly connected graph, also called **covers**.

Complete deterministic local automata were called definite automata [Perles, Rabin, Shamir 63].

An (m, a) -Local automaton. m memory, a anticipation

An automaton is (m, a) -local if whenever two paths of length $m + a$ have the same label, they go through the same state at time m .

The minimal length $m + a$ such that the automaton fits this condition is called the **synchronization delay** of the automaton.



The cover

- is $(2, 1)$ -local
- has a synchronization delay 3
- is complete

Proposition (Perles, Rabin, Shamir, 63)

An n -state deterministic local complete automaton has a synchronization delay at most $n - 1$.

Proposition

When the local automaton is not complete, the upper bound of the synchronization delay is quadratic in the number of states.

Theorem (Czeiler, Kari, ICALP'05)

An n -state local complete automaton has a synchronization delay at most $n - 1$.

We present here a new proof of this result.

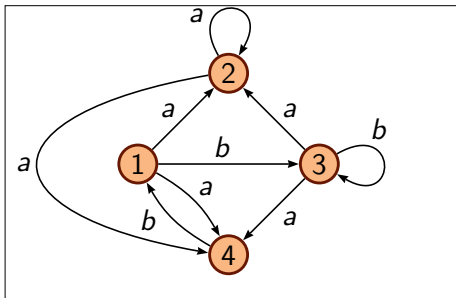
Definition

Let Q be a finite set. A monoid of unambiguous relations M is a submonoid of $\{0, 1\}^{Q \times Q}$. The coefficients 0, 1 of an element belong to \mathbb{N} , and not to the Boolean semiring.

- M is **complete** if does not contain the null matrix.
- M is **transitive** if for any $p, q \in Q$, there is $m \in M$ such that $M_{pq} = 1$.
- M is **synchronizing** if it contains a matrix of rank 1.

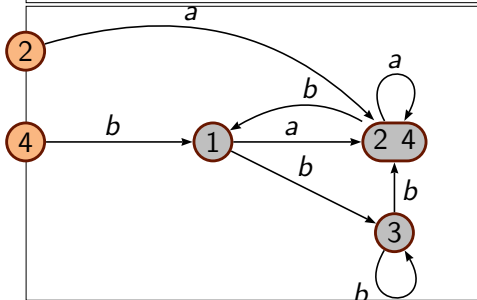
A maximal row is the row of an element of M maximal for the inclusion. We denote by R (resp. C) the set of maximal rows (resp. columns).

Example. Transition monoid of a complete transitive cover



The cover \mathcal{A}

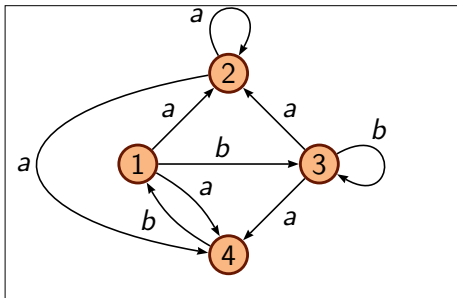
$$\mu(aa) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



The automaton $\text{det}(\mathcal{A})$ starting from singletons.

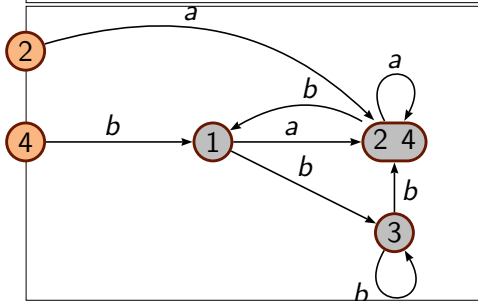
The maximal rows correspond to the states 1, 3 and 2, 4.

Example. Transition monoid of a complete transitive cover



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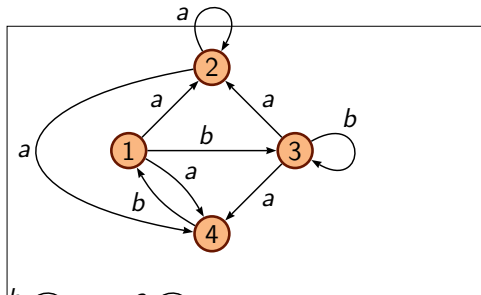
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Proposition (Césari, Boë)

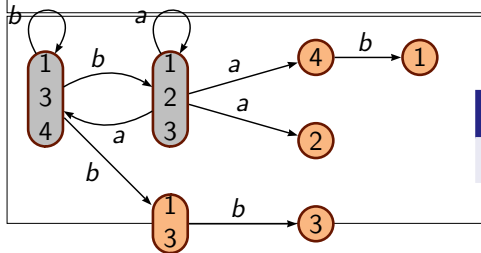
$$RM = R.$$

Example. Transition monoid of a complete transitive cover



The cover \mathcal{A}

$$\mu(ab) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



The automaton $\text{codet}(\mathcal{A})$

Proposition (Césari, Boë)

$$MC=C.$$

A pair (R, C) is called a **box** of Q if and only if

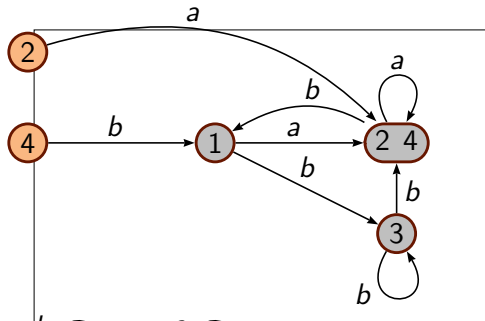
- 1 R is a set of Q -zero-one row vectors covering Q .
- 2 C is a set of Q -zero-one column vectors covering Q .
- 3 for any vectors $r \in R, c \in C, r \cdot c = 1$.

Let M be a complete monoid of unambiguous relations with maximal rows and columns R, C .

Proposition (one side from Boë)

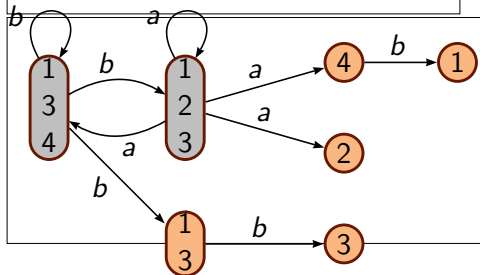
The pair (R, C) is a box if and only M is synchronizing.

Boxes. Example



(R, C) is a box

$R \setminus C$	123	134
1	1	1
24	2	4
3	3	3



A new proof of

An n -state local complete automaton has a synchronization delay at most $n - 1$.

- Let $D_i = \langle \{ru - r'u\} \mid r, r' \in R, u \in A^i \rangle$.

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As a consequence, for any $u \in A^m$, any $r, r' \in R$, $ru - r'u = 0$.
- $D_0 \supseteq D_1 \supseteq \dots \supseteq D_i \supseteq D_{i+1} \supseteq \dots \supseteq D_m = \{0\}$.
- If $D_i = D_{i+1}$, then $D_i = D_{i+1} = \dots = D_m = \{0\}$.
 $\Rightarrow m \leq \dim D_0$

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- We have $E_0 \subsetneq \langle C \rangle$.
- From $(r - r') \cdot c = 0$ for any $r, r' \in R$, $c \in C$, we get $\dim D_0 + \dim C \leq \text{Card } Q$.

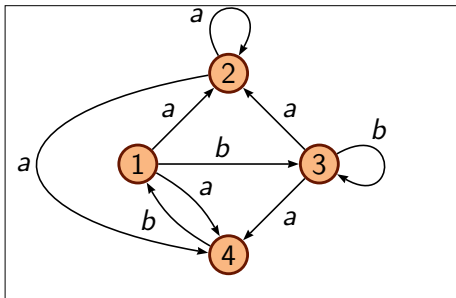
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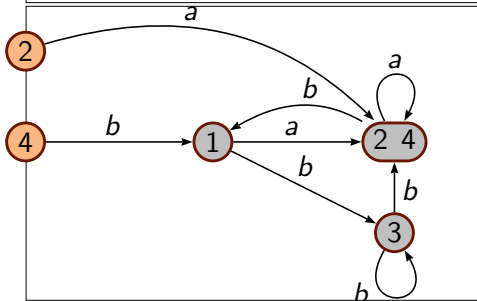
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- $m \leq \dim D_0$ and $a \leq \dim E_0$.
- We have $E_0 \subsetneq \langle C \rangle$.
- From $(r - r') \cdot c = 0$ for any $r, r' \in R$, $c \in C$, we get $\dim D_0 + \dim C \leq \text{Card } Q$.
- Finally, we obtain

$$\begin{aligned} m + a &\leq \dim D_0 + \dim E_0, \\ &\leq \dim D_0 + \dim \langle C \rangle - 1, \\ &\leq \text{Card } Q - 1. \end{aligned}$$

Example



The cover \mathcal{A} is $(2, 1)$ -local.



The vectorial space D_0

- is generated by $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$.
- has dimension 2.

Definition

If w is a right-infinite word, the Welch set $R(w) = \{q \mid \exists \xrightarrow{w} q\}$.
If w is a left-infinite word, the Welch set $L(w) = \{q \mid \exists q \xrightarrow{w}\}$.

Proposition

One has $R = \bigcup_w R(w)$ and $C = \bigcup_w C(w)$.